Supplement to "Cotrending: testing for common deterministic trends in varying means model"

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We present here a supplementary result for the article "Cotrending: testing for common deterministic trends in varying means model" which is crucial to prove Proposition 4. We adopt the notation of the article and refer to its labels.

B. Supplement

Lemma B.1. *Set* $Z_{1,t} = \sum_{i=1}^{t} \varepsilon_i$ *and* $Z_{2,t} = \sum_{j=0}^{\infty} \widetilde{L}_j \varepsilon_{t-j}$, *where* $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ *is a sequence of i.i.d. random vectors satisfying* (28) *with positive definite* Σ_ε *and* $\mathrm{E}\,\|\varepsilon_0\|^4 < \infty$, *and* $\sum_{j=0}^{\infty} \|\widetilde{L}_j\|_F < \infty$. *Then,*

(i) $\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t-1}\varepsilon_t^\top \xrightarrow{\mathrm{d}} \frac{1}{2}(\Sigma_\varepsilon^{\frac{1}{2}} Z(1)Z(1)^\top \Sigma_\varepsilon^{\frac{1}{2}} - \Sigma_\varepsilon)$,

(ii) $\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t}Z_{2,t+1}^\top \xrightarrow{\mathrm{d}} \frac{1}{2}(\Sigma_\varepsilon^{\frac{1}{2}} Z(1)Z(1)^\top \Sigma_\varepsilon^{\frac{1}{2}} + \Sigma_\varepsilon)\sum_{j=0}^{\infty} \widetilde{L}_j^\top$,

(iii) $\frac{1}{T^{3/2}}\sum_{t=1}^{T} Z_{1,t} \xrightarrow{\mathrm{d}} \Sigma_\varepsilon^{1/2}\int_0^1 Z(t)dt$,

(iv) $\frac{1}{T}\sum_{t=1}^{T-1} Z_{2,t}Z_{2,t+1}^\top - \mathrm{Cov}(Z_{2,0}, Z_{2,1}) \xrightarrow{\mathrm{p}} 0$,

where $Z(t)$ *is a* p*-dimensional standard Brownian motion.*

Proof: The statement (i) is the same as in Lemma 3.1, (d) in Phillips and Durlauf [3].

For the convergence in (ii), set

$$\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t}Z_{2,t+1}^\top = \sum_{j=0}^{\infty}\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t}\varepsilon_{t-(j-1}^\top \widetilde{L}_j^\top =: \sum_{j=0}^{\infty} Y_j(T).$$

Then, by Theorem 4.2 in Billingsley [1], it is enough to prove

$$\sum_{j=0}^{k} Y_j(T) \xrightarrow{\mathrm{d}} \frac{1}{2}(\Sigma_\varepsilon^{\frac{1}{2}} Z(1)Z(1)^\top \Sigma_\varepsilon^{\frac{1}{2}} + \Sigma_\varepsilon)\sum_{j=0}^{k} \widetilde{L}_j^\top \tag{B.1}$$

for each $k \geqslant 1$ and

$$\sum_{j=k+1}^{\infty} Y_j(T) = o_{\mathrm{p}}(1), \quad \text{as } T \to \infty, \ \ k \to \infty. \tag{B.2}$$

The convergence in (B.1) is a consequence of

$$\sum_{j=0}^{k} Y_j(T) = \sum_{j=0}^{k}\frac{1}{T}\sum_{t=1}^{T-1}\Big(Z_{1,t-j}\varepsilon_{t-(j-1)}^\top + \sum_{i=t-j+1}^{t}\varepsilon_i\varepsilon_{t-(j-1)}^\top\Big)\widetilde{L}_j^\top$$

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$$
\begin{aligned}
&= \sum_{j=0}^{k}\Big(\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t-j}\varepsilon_{t-(j-1)}^{\top} + \sum_{l=0}^{j-1}\frac{1}{T}\sum_{t=1}^{T-1}\varepsilon_{t-l}\varepsilon_{t-(j-1)}^{\top}\Big)\widetilde{L}_j^{\top} \\
&= \sum_{j=0}^{k}\Big(\frac{1}{T}\sum_{t=1-j}^{T-1-j} Z_{1,t}\varepsilon_{t+1}^{\top} + \Sigma_\varepsilon\Big)\widetilde{L}_j^{\top} + o_{\mathrm{p}}(1) && \text{(B.3)}\\
&= \Big(\frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t}\varepsilon_{t+1}^{\top} + \Sigma_\varepsilon\Big)\sum_{j=0}^{k}\widetilde{L}_j^{\top} + o_{\mathrm{p}}(1) && \text{(B.4)}\\
&\xrightarrow{\mathrm{d}} \frac{1}{2}(\Sigma_\varepsilon^{\frac{1}{2}}Z(1)Z(1)^{\top}\Sigma_\varepsilon^{\frac{1}{2}} + \Sigma_\varepsilon)\sum_{j=0}^{k}\widetilde{L}_j^{\top}. && \text{(B.5)}
\end{aligned}
$$

The equality (B.3) follows since $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is stationary and ergodic, and so is any transformation of ε_j. Indeed, by the ergodic theorem and since $\mathrm{E}\,\varepsilon_{t-l}\varepsilon_{t-(j-1)}^{\top} = \Sigma_\varepsilon$ for $l = j-1$ and 0 otherwise,

$$
\frac{1}{T}\sum_{t=1}^{T-1}\sum_{l=0}^{j-1}\varepsilon_{t-l}\varepsilon_{t-(j-1)}^{\top}\widetilde{L}_j^{\top} = \mathrm{E}\Big(\sum_{l=0}^{j-1}\varepsilon_{t-l}\varepsilon_{t-(j-1)}^{\top}\widetilde{L}_j^{\top}\Big) + o_{\mathrm{p}}(1) = \Sigma_\varepsilon\widetilde{L}_j^{\top} + o_{\mathrm{p}}(1); \tag{B.6}
$$

see Theorem 2 in Hannan [2], p. 203. For the equality (B.4) note that

$$
\frac{1}{T}\sum_{t=1-j}^{T-1-j} Z_{1,t}\varepsilon_{t+1}^{\top} = \frac{1}{T}\sum_{t=1}^{T-1} Z_{1,t}\varepsilon_{t+1}^{\top} + o_{\mathrm{p}}(1), \tag{B.7}
$$

since

$$
\mathrm{E}\,\|\frac{1}{T}\sum_{t=r}^{s} Z_{1,t}\varepsilon_{t+1}^{\top}\|_F^2 = \frac{1}{T^2}\sum_{t_1,t_2=r}^{s}\sum_{i_1=1}^{t_1}\sum_{i_2=1}^{t_2}\mathrm{E}\,\mathrm{tr}(\varepsilon_{t_1+1}\varepsilon_{i_1}^{\top}\varepsilon_{i_2}\varepsilon_{t_2+1}^{\top}) = \frac{1}{T^2}\sum_{t=r}^{s} t(\mathrm{E}\,\|\varepsilon_0\|^2)^2 = o(1), \tag{B.8}
$$

where either $r = 1-j$ and $s = 0$ or $r = T-j$ and $s = T-1$. The convergence in (B.5) is a consequence of (i).

The equality (B.2) can be proven by

$$
\begin{aligned}
&\mathrm{E}\,\|\sum_{j=k+1}^{\infty} Y_j(T)\|_F^2 = \sum_{j_1,j_2=k+1}^{\infty}\frac{1}{T^2}\sum_{t_1,t_2=1}^{T-1}\sum_{i_1=1}^{t_1}\sum_{i_2=1}^{t_2}\mathrm{E}\,\mathrm{tr}\Big(\widetilde{L}_{j_1}\varepsilon_{t_1-(j_1-1)}\varepsilon_{i_1}^{\top}\varepsilon_{i_2}\varepsilon_{t_2-(j_2-1)}^{\top}\widetilde{L}_{j_2}^{\top}\Big)\\
&= \sum_{j_1,j_2=k+1}^{\infty}\frac{1}{T^2}\sum_{l_1=2-j_1}^{T-j_1}\sum_{l_2=2-j_2}^{T-j_2}\sum_{i_1=1}^{l_1+j_1-1}\sum_{i_2=1}^{l_2+j_2-1}\mathrm{tr}\Big(\mathrm{E}(\varepsilon_{l_1}\varepsilon_{i_1}^{\top}\varepsilon_{i_2}\varepsilon_{l_2}^{\top})\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1}\Big)\\
&= \sum_{j_1,j_2=k+1}^{T+1}\frac{1}{T^2}\sum_{l=2-\overline{m}}^{T-\underline{m}}\mathrm{tr}(\Sigma^*\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1}) + 2\sum_{j_1,j_2=k+1}^{T+1}\frac{1}{T^2}\sum_{l_1=2-j_1}^{T-j_1}\sum_{l_2=2-j_2}^{T-j_2}\mathrm{tr}(\Sigma_\varepsilon^2\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})\\
&\quad + \sum_{j_1,j_2=k+1}^{\infty}\frac{1}{T^2}\sum_{l=2-\overline{m}}^{T-\underline{m}}\sum_{i=1}^{l+\underline{m}-1}\mathrm{E}\,\|\varepsilon_0\|^2\,\mathrm{tr}(\Sigma_\varepsilon\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})\\
&= \sum_{j_1,j_2=k+1}^{T+1}\frac{1}{T^2}(T-1+\overline{m}-\underline{m})(\mathrm{tr}(\Sigma^*\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1}) + \mathrm{E}\,\|\varepsilon_0\|^2\,\mathrm{tr}(\Sigma_\varepsilon\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})) + 2\sum_{j_1,j_2=k+1}^{T+1}\frac{1}{T^2}(T-1)^2\,\mathrm{tr}(\Sigma_\varepsilon^2\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})\\
&\leqslant 2\sum_{j_1,j_2=k+1}^{T+1}\frac{1}{T}|\,\mathrm{tr}(\Sigma^*\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1}) + \mathrm{E}\,\|\varepsilon_0\|^2\,\mathrm{tr}(\Sigma_\varepsilon\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})| + 2\sum_{j_1,j_2=k+1}^{T+1}|\,\mathrm{tr}(\Sigma_\varepsilon^2\widetilde{L}_{j_2}^{\top}\widetilde{L}_{j_1})| \to 0,
\end{aligned}
$$

as $T\to\infty$ and $k\to\infty$, since $\sum_{j=0}^{\infty}\|\widetilde{L}_j\|_F < \infty$. Thereby, we used the notation $\underline{m} = \min\{j_1, j_2\}$, $\overline{m} = \max\{j_1, j_2\}$,

$\Sigma^* := \mathrm{E}(\varepsilon_0\varepsilon_0^\top\varepsilon_0\varepsilon_0^\top)$ and the fact that

$$\mathrm{E}(\varepsilon_{l_1}\varepsilon_{i_1}^\top\varepsilon_{i_2}\varepsilon_{l_2}^\top) = \begin{cases} \Sigma^*, & i_1 = i_2 = l_1 = l_2, \\ \Sigma_\varepsilon^2, & i_1 = l_1 \neq i_2 = l_2, \\ \Sigma_\varepsilon^2, & i_1 = l_2 \neq i_2 = l_1, \\ \mathrm{E}\,\|\varepsilon_0\|^2\Sigma_\varepsilon, & i_1 = i_2 \neq l_1 = l_2. \end{cases}$$

The statement (iii) is proven in Lemma 3.1, (a) in Phillips and Durlauf [3], p. 210. The last point (iv) gives the weak law of large numbers for the sample autocovariances of linear processes and is proven in Hannan [2], p. 210. □

References

[1] P. Billingsley, Probability and Measure, John Wiley and Sons, New York, second edition, 1986.

[2] E. J. Hannan, Multiple Time Series, Wiley Series in Probability and Statistics, Wiley, New York, 1970.

[3] P. C. B. Phillips, S. N. Durlauf, Multiple time series regression with integrated processes, The Review of Economic Studies 53 (1986) 473–495.