Sample autocovariance operators of Hilbert space-valued linear processes

Marie-Christine Düker*

Cornell University, Department of Statistics and Data Science

January 26, 2021

Abstract

This article considers linear processes in a separable Hilbert space under long-range dependence. A central limit theorem for the sample autocovariance operators at different time lags in the space of Hilbert-Schmidt operators is investigated. The limiting process follows either a Hilbert space-valued Gaussian process or a non-Gaussian process, represented in terms of stochastic integrals. The key technical result is the introduction of double Wiener-Itô integrals with values in a function space, in order to represent the limiting process of the sample autocovariance operators.

Keywords: Hilbert space, stochastic integrals, linear processes, autocovariance operators, long-range dependence.

1 Introduction

In this article, we are interested in the asymptotic behavior of the sample autocovariance operators of a Hilbert space-valued linear process under long-range dependence. The investigation is related to functional data analysis. For this reason, the respective Hilbert space is often a function space. We refer to Ramsay and Silverman (1997); Ferraty and Vieu (2006); Horváth and Kokoszka (2012) for more details about functional data analysis and to Bosq (2000) for insights about Hilbert space-valued linear processes.

Our setting is as follows: Let $\mathbb{H}$ denote a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We further write $L(\mathbb{H})$ for the set of all bounded linear operators on $\mathbb{H}$. We consider a sequence of random variables $\{X_n\}_{n \in \mathbb{Z}}$ defined on a general probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{H}$. Suppose the time series $\{X_n\}_{n \in \mathbb{Z}}$ has a linear representation

$$X_n = \sum_{j \in \mathbb{Z}} u_j(\varepsilon_{n-j}), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\{u_j\}_{j \in \mathbb{Z}}$ is a sequence of linear operators $u_j : \mathbb{H} \to \mathbb{H}$ and $\{\varepsilon_j\}_{j \in \mathbb{Z}}$ is a sequence of $\mathbb{H}$-valued independent, identically distributed (i.i.d.), zero mean random variables. We write the operators u_j in (1.1) as

$$u_j = (j+1)^{T-I}, \quad j \geqslant 0, \tag{1.2}$$

where I denotes the identity operator, $T \in L(\mathbb{H})$ is a self-adjoint operator, that is, $T = T^*$, where T^* denotes the Hermitian adjoint corresponding to T. Furthermore, we suppose that the

*Cornell University, Department of Statistics and Data Science, Ithaca, New York 14853, md2224@cornell.edu

series over the operator norm of the u_j diverges, that is,

$$\sum_{j=0}^{\infty} \|u_j\|_{op} = \infty$$

to impose long-range dependence.

We are interested here in the asymptotic behavior of the sample autocovariance operators at different time lags ℓ, defined as

$$((\widehat{\Gamma}_{N,\ell} - \Gamma_\ell), \ell = 0, \dots, L) \tag{1.3}$$

with

$$\widehat{\Gamma}_{N,\ell} = \frac{1}{N} \sum_{n=1}^{N} \langle X_{n+\ell}, \cdot \rangle X_n \quad \text{and} \quad \Gamma_\ell = \mathrm{E}(\langle X_\ell, \cdot \rangle X_0).$$

The sample autocovariance operators are considered to be random elements with values in the space of Hilbert-Schmidt operators denoted by $\mathcal{S}$. A Hilbert-Schmidt operator $A : \mathbb{H} \to \mathbb{H}$ is a bounded operator with finite Hilbert-Schmidt norm

$$\|A\|_{\mathcal{S}} = \left(\sum_{i=1}^{\infty} \|Ae_i\| \right)^{\frac{1}{2}} < \infty.$$

The space $\mathcal{S}$ equipped with the inner product $\langle A, B \rangle_{\mathcal{S}} = \sum_{i=1}^{\infty} \langle Ae_i, Be_i \rangle$ and the norm $\|A\|_{\mathcal{S}}$ is a separable Hilbert space itself.

Our work is closely related to functional data analysis which has been studied extensively over the last couple of years in various contexts; see Hörmann, Kokoszka, and Nisol (2018); Aneiros, Cao, Fraiman, Genest, and Vieu (2019); Dette, Kokot, and Aue (2020); van Delft and Dette (2021). However, the related literature is mostly restricted to functional data under short-range dependence. A similar restriction can be observed in a rather theoretical series of works (Merlevède, Peligrad, and Utev, 1997; Mas, 2002; Jirak, 2018) which investigated the asymptotic behavior of Hilbert space-valued linear processes under short-range dependence.

Long-range dependence has been studied extensively for real-valued time series; see Giraitis, Koul, and Surgailis (2012); Beran, Feng, Ghosh, and Kulik (2013); Pipiras and Taqqu (2017) for comprehensive collections on this topic. The study of long-range dependent time series entails in particular non-central limit theorems which lead to the so-called Rosenblatt process; see Taqqu (1975); Rosenblatt (1979); Fox and Taqqu (1985). The interest in long-range dependence is continuous and its exploration is ongoing (Pipiras and Taqqu, 2003; Didier and Pipiras, 2011; Leonenko, Ruiz-Medina, and Taqqu, 2017) as well as the study of non-central limit theorems and the Rosenblatt process; see Tudor (2008); Veillette and Taqqu (2013); Yan, Li, and Wu (2015); Bai and Taqqu (2017); Anh, Leonenko, Olenko, and Vaskovych (2019); Bai and Taqqu (2020). Our exploration here will lead to the introduction of a Hilbert space-valued version of the Rosenblatt process.

Sample autocovariances were studied by numerous authors. For real-valued Gaussian time series under long-range dependence Taqqu (1975) and Taqqu (1979) proved the weak convergence of the sample autocovariance process to the so-called Rosenblatt process. In Horváth and Kokoszka (2008), this result was generalized to linear processes under long-range dependence. Düker (2020) derived a functional central limit for the sample autocovariance matrices of multivariate linear processes under quite general assumptions on the dependence structure including multivariate long-range dependence.

Sample autocovariance operators for Hilbert space-valued processes were studied by Mas (2002), who proved a limit theorem for the sample autocovariance operators (1.3) of a short-range dependent linear process. Mas (2002) imposed short-range dependence by supposing a representation (1.1) with $\sum_{j=0}^{\infty} \|u_j\|_{op} < \infty$.

Hilbert space-valued linear processes under long-range dependence were considered by Characiejus and Račkauskas (2013) and Characiejus and Račkauskas (2014). They assumed (1.1) to take values in the Hilbert space of square-integrable real-valued functions $L^2(\mu) = L^2_{\mathbb{R}}(\mathbb{S}, \mathcal{S}, \mu)$, where $(\mathbb{S}, \mathcal{S}, \mu)$ is a σ-finite measure space. One can get a representation with values in $L^2(\mu)$ by choosing

$$u_j = (j+1)^{D-I} \tag{1.4}$$

in (1.1), where D is a multiplication operator given by $Df = \{d(s)f(s) | s \in \mathbb{S}\}$ for each $f \in L^2(\mu)$ and a measurable function $d : \mathbb{S} \to \mathbb{R}$. Characiejus and Račkauskas (2013, 2014) proved weak convergence of the sample mean process to a Hilbert space-valued fractional Brownian motion. Their results were generalized by Düker (2018) to processes with representation (1.1) with (1.2) and values in a possible complex separable Hilbert space.

The work in Mas (2002) is probably the closest to our study. However, the behavior of the linear process $\{X_n\}_{n\in\mathbb{Z}}$ crucially depends on the convergence behavior of the series $\sum_{j=0}^{\infty} \|u_j\|_{op}$. Mas (2002) assumed that $\sum_{j=0}^{\infty} \|u_j\|_{op} < \infty$, which allows for the normalization sequence $N^{\frac{1}{2}}$, this is the same normalization sequence as for i.i.d. data. Furthermore, in Mas (2002), the normalized sample mean converges to a Hilbert space-valued Gaussian random element. In contrast, our setting requires a normalization sequence in terms of the self-adjoint operator T in (1.2) and the limit is possibly non-Gaussian. The non-Gaussian limit requires a proper representation. For this purpose, we introduce double Wiener-Itô integrals with values in a function space. The introduction of double Wiener-Itô integrals with values in a function space allows us to give a Hilbert space-valued version of the so-called Rosenblatt process which is well studied as a real-valued object; see Taqqu (1975); Rosenblatt (1979).

The rest of the paper is organized as follows. In Section 2, we introduce some notation and give preliminary results. In Section 3, we present our main result with sufficient conditions for the convergence of the sample autocovariance operators of the process $\{X_n\}_{n\in\mathbb{Z}}$. In Section 4, we introduce double Wiener-Itô integrals with values in a function space, which is crucial to represent the resulting limit. In Section 5, we present the proofs of our main results and the last Section 6 is concerned with some technical results and their proofs.

2 Preliminaries

In this section, we introduce some notation, a class of function spaces, the spectral theorem for self-adjoint operators and give some properties of the linear process (1.1).

The spectral theorem for self-adjoint operators states that each self-adjoint operator $T \in L(\mathbb{H})$ is decomposable into a unitary operator $U : \mathbb{H} \to L^2(\mu)$ and a multiplication operator $D : L^2(\mu) \to L^2(\mu)$; see Theorem 9.4.6 in Comway (1994). More precisely, there exist a σ-finite measure space $(\mathbb{S}, \mathcal{S}, \mu)$ and a unitary operator $U : \mathbb{H} \to L^2(\mu)$ together with a bounded function $d : \mathbb{S} \to \mathbb{R}$, such that

$$UTU^* = D, \tag{2.1}$$

where D is a multiplication operator given by $Df = \{d(s)f(s) | s \in \mathbb{S}\}$ for each $f \in L^2(\mu)$. The spectral theorem for self-adjoint operators allows us to express the convergence result for the sample autocovariance operators in terms of the sample autocovariances of an $L^2(\mu)$-valued linear process. For this reason, we introduce some notation in terms of an $L^2(\mu)$-valued linear process. Suppose the underlying linear process (1.1) takes values in $L^2(\mu)$ with

$$X_n = \sum_{j=0}^{\infty} (j+1)^{d(r)-1} \varepsilon_{n-j}(r), \tag{2.2}$$

where $\{\varepsilon_j(r)\}_{j\in\mathbb{Z}}$ is an $L^2(\mu)$-valued i.i.d. sequence with

$$\sigma(r,s) := E(\varepsilon_0(r)\varepsilon_0(s)), \quad \sigma^2(r) := E|\varepsilon_0(r)|^2 \quad r,s \in \mathbb{S}. \tag{2.3}$$

For the sample autocoariances of $\{X_n\}_{n\in\mathbb{Z}}$ in (2.2) and its corresponding population quantities, we write

$$\widehat{\gamma}_{N,\ell}(r,s) := \frac{1}{N}\sum_{n=1}^{N} X_n(r)X_{n+\ell}(s) \quad \text{and} \quad \gamma_\ell(r,s) := E(X_0(r)X_\ell(s)). \tag{2.4}$$

It is well known that the so-called beta function is a function of two complex numbers a, b with positive real part defined by $\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$. It may be also written as $\mathrm{B}(a,b) = \int_0^\infty x^{a-1}(x+1)^{-(a+b)}dx$. We define the function $c : \mathbb{S}\times\mathbb{S} \to \mathbb{R}$ by

$$c(r,s) = \mathrm{B}(d(r),d(s)) = \int_0^\infty x^{d(r)-1}(x+1)^{d(s)-1}dx. \tag{2.5}$$

For shortness' sake, we introduce a class of function spaces. Let $(X,\mathcal{A},\alpha)$ be a measure space and $(Y,\|\cdot\|_Y)$ a normed space, then $L(X,Y)$ denotes the space of bounded linear functions from X to Y and $L^2_Y(X,\mathcal{A},\alpha)$ the space of square-integrable Y-valued measurable functions. Then, the space $L^2_Y(X,\mathcal{A},\alpha)$ equipped with the norm

$$\|\cdot\|_{L^2} : L(X,Y) \to \mathbb{R} \quad \text{with} \quad \|f\|_{L^2} = \int_X \|f\|_Y^2 d\alpha$$

is a Hilbert space. Furthermore, we introduce some abbreviations for the function spaces used throughout this article. We already met the space of square-integrable real-valued functions, abbreviated as $L^2(\mu) := L^2_{\mathbb{R}}(\mathbb{S},S,\mu)$. Note that the inner product of $L^2(\mu)$ is given by

$$\langle f,g\rangle_{L^2(\mu)} = \int_{\mathbb{S}} f(s)g(s)\mu(ds), \quad f,g \in L^2(\mu).$$

For the tensor product of $L^2(\mu)$, we write

$$L^2(\mu\otimes\mu) := L^2(\mu)\otimes L^2(\mu) \cong L^2_{\mathbb{R}}(\mathbb{S}\times\mathbb{S}, S\times S, \mu\otimes\mu),$$

where $\cong$ denotes an isomorphism. The same way, we write $L^2(\nu) := L^2_{\mathbb{R}}(\mathbb{S},S,\nu)$, where the measure ν is defined by

$$\nu(A) = \int_A \sigma^2(r)\mu(dr) \quad \text{for} \quad A \in S$$

with $\sigma^2(r) = \mathrm{E}\,|\varepsilon_0(r)|^2$ and the corresponding tensor product

$$L^2(\nu\otimes\nu) = L^2(\nu)\otimes L^2(\nu) \cong L^2_{\mathbb{R}}(\mathbb{S}\times\mathbb{S}, S\times S, \nu\otimes\nu).$$

Further spaces of such type are $L^2_{L^2(\mu\otimes\mu)}(\mathbb{R}^2) := L^2_{L^2(\mu\otimes\mu)}(\mathbb{R}^2,\mathcal{B}(\mathbb{R}^2),\lambda^2)$, where λ^2 denotes the two-dimensional Lebesgue measure and $L^2_{L^2(\mu\otimes\mu)}(\Omega) := L^2_{L^2(\mu\otimes\mu)}(\Omega,\mathcal{F},P)$. The space $L^2_{L^2(\mu\otimes\mu)}(\Omega)$ contains functions on a probability space $(\Omega,\mathcal{F},P)$.

Due to the spectral theorem, which gives the relation (2.1), deriving a limit theorem for the sample autocovariance operators of a Hilbert space-value linear process reduces to finding the limit for a linear process with value in $L^2(\mu)$ representable as in (2.2). However, we present both, since the spectral theorem allows us to find an explicit form for the limit of the sample autocovariance operators of a linear process with value in a general Hilbert space $\mathbb{H}$. The unitary operator

$$F : L^2(\mu \otimes \mu) \to \mathcal{S}, \ \ A(r,s) \mapsto U^* \left(\int_{\mathbb{S}} A(r,s) U((\cdot)(r)) \mu(dr) \right) \tag{2.6}$$

is crucial in finding an explicit representation for the limits in our main result.

We conclude this section with some properties of the linear process (1.1) with (1.2) and its sample autocovariance operators (1.3). However, the proofs of those properties are postponed to Section 5.1 as part of the proof of our main result. The series (1.1) with u_j as in (1.2) converges P-almost surely, where P denotes the probability measure of the probability space our random variables are defined on. The sample autocovariance operators (1.3) belong P-almost surely to the space of Hilbert-Schmidt operators $\mathcal{S}$. To impose long-range dependence on the process $\{X_n\}_{n\in\mathbb{Z}}$, we suppose that the series of operator norms $\sum_{j=0}^{\infty} \|(j+1)^{T-I}\|_{op}$ diverges. This is the case if and only if $\operatorname{ess\,sup}_{s\in\mathbb{S}} d(s) \geqslant 0$. The function $d : \mathbb{S} \to \mathbb{R}$ defines the multiplication operator which decomposes the self-adjoint operator T using the spectral representation (2.1).

3 Main Results

In this section, we introduce the limits and state the convergence result for the sample autocovariance operators (1.3).

From the convergence result for the sample autocovariances of real-valued long-range dependent processes in Horváth and Kokoszka (2008), it is well-known that the limit follows a Brownian motion when the long-range dependence parameter of the underlying linear process takes values in $(0, \frac{1}{4})$, and is the Rosenblatt process evaluated in one when the long-range dependence parameter takes values in $(\frac{1}{4}, \frac{1}{2})$. This suggests to distinguish similar cases in our context of Hilbert space-valued linear processes. However, instead of distinguishing those cases for the operator T itself, we use the decomposition (2.1) and distinguish $d(s) \in (0, \frac{1}{4})$ and $d(s) \in (\frac{1}{4}, \frac{1}{2})$ for each $s \in \mathbb{S}$.

In the first case, when $d(s) \in (0, \frac{1}{4})$, the limit $\mathcal{G}_\ell$ is Gaussian with cross-covariance operator $C_{\mathcal{G},F} : \mathcal{S} \to \mathcal{S}$ given by

$$C_{\mathcal{G},F}(x) = \mathrm{E}(\langle \mathcal{G}_{\ell_1}, x \rangle_{\mathcal{S}} \mathcal{G}_{\ell_2}) = F \langle A_{U,\ell_1,\ell_2}, F^*(x) \rangle_{L^2(\mu\otimes\mu)}, \ \ x \in \mathcal{S}, \tag{3.1}$$

with $F : L^2(\mu \otimes \mu) \to \mathcal{S}$ as in (2.6) and

$$\begin{aligned} &A_{U,\ell_1,\ell_2}(r_1, s_1, r_2, s_2) \\ &= \sum_{r\in\mathbb{Z}} (\gamma_{U,r}(r_1, r_2)\gamma_{U,\ell_2-\ell_1+r}(s_1, s_2) + \gamma_{U,r+\ell_2}(r_1, s_2)\gamma_{U,r-\ell_1}(s_1, r_2)) \\ &\qquad + \sum_{r\in\mathbb{Z}} \sum_{i=0}^{\infty} u_i(r_1)u_{i+\ell_1}(s_1)u_{i+r}(r_2)u_{i+\ell_2+r}(s_2)\Sigma_U(r_1, s_1, r_2, s_2), \end{aligned} \tag{3.2}$$

where

$$u_j(s) = \begin{cases} (j+1)^{d(s)-1}, & \text{if } j \geqslant 0, \\ 0, & \text{if } j < 0 \end{cases} \tag{3.3}$$

and

$$\begin{aligned} \Sigma_U(r_1, s_1, r_2, s_2) = \sigma_U^*(r_1, s_1, r_2, s_2) &- \sigma_U(r_1, s_1)\sigma_U(r_2, s_2) \\ &- \sigma_U(r_1, r_2)\sigma_U(s_1, s_2) - \sigma_U(r_1, s_2)\sigma_U(s_1, r_2) \end{aligned} \tag{3.4}$$

with

$$\sigma_U^*(r_1,s_1,r_2,s_2) := \mathrm{E}(U\varepsilon(r_1)U\varepsilon(s_1)U\varepsilon(r_2)U\varepsilon(s_2)), \quad \sigma_U(r,s) := \mathrm{E}(U\varepsilon(r)U\varepsilon(s)) \tag{3.5}$$

and

$$\gamma_{U,\ell}(r,s) = \sum_{j=0}^{\infty} u_j(r)u_{j+\ell}(s)\sigma_U(r,s). \tag{3.6}$$

In the second case, when $d(s) \in (\frac{1}{4}, \frac{1}{2})$ for $s \in \mathbb{S}$, the limit is no longer Gaussian. The resulting limit process is a generalization of the so-called Rosenblatt process evaluated in one. However, the limit here takes values in the space of Hilbert-Schmidt operators. We will represent it by means of double Wiener-Itô integrals with values in the function space $L^2(\nu \otimes \nu)$. The measure ν on S is defined in terms of $\sigma_U^2(r) = \sigma_U(r,r)$ as

$$\nu(A) = \int_A \sigma_U^2(r)\mu(dr) \quad \text{for} \quad A \in S$$

with $\sigma_U(r,r)$ as in (3.5). Double Wiener-Itô integrals with values in function spaces are properly introduced and defined in Section 4 below. Let $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$ denote the space of all functions $f : \mathbb{R}^2 \to L^2(\nu \otimes \nu)$ equipped with the norm

$$\|f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \|f(x_1,x_2)\|^2_{L^2(\nu\otimes\nu)} dx_1 dx_2 < \infty.$$

Then, for $f \in L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$, a double Wiener-Itô integral with respect to an $L^2(\mu)$-valued Gaussian random measure $W_U^{(r)}(dx)$ with $\mathrm{E}\, W_U^{(r)}(dx)W_U^{(s)}(dx) = \sigma_U(r,s)dx$ is defined as

$$I_2(f^{(r,s)}) = \int_{\mathbb{R}^2}' f^{(r,s)}(x_1,x_2)W_U^{(r)}(dx_1)W_U^{(s)}(dx_2), \tag{3.7}$$

where $\int_{\mathbb{R}^2}'$ means that integration excludes the diagonals, see Definition 4.2. Our future limit, the $\mathbb{H}$-valued random element Z is defined as

$$Z = F(\widetilde{Z}_U) \quad \text{with} \quad \widetilde{Z}_U(r,s) = I_2(f^{(r,s)}) \tag{3.8}$$

with $I_2(f^{(r,s)})$ as in (3.7), $F : L^2(\nu \otimes \nu) \to \mathcal{S}$ as in (2.6) and

$$f^{(r,s)}(x_1,x_2) := \int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1} dv. \tag{3.9}$$

Note that $x_+ = \max\{0,x\}$. The random variable $\widetilde{Z}$ in (3.8) takes values in $L^2(\nu \otimes \nu)$ and f in (3.9) takes values in $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$, see Lemma 6.5 in Section 6.

The following theorem gives the limit of the sample autocovariance operators. Throughout the paper, we write $\xrightarrow{\mathrm{d}}$ for convergence in distribution.

Theorem 3.1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a stationary linear process (1.1) with (1.2) and $\mathrm{E}\,\|\varepsilon_0\|^4 < \infty$.*

(i) If $d(s) \in (0, \frac{1}{4})$ for each $s \in \mathbb{S}$ and

$$\int_{\mathbb{S}} \frac{(\mathrm{E}(U\varepsilon(s))^4)^{\frac{1}{2}}}{d(s)^2}\mu(ds) < \infty, \tag{3.10}$$

then,

$$(N^{\frac{1}{2}}(\widehat{\Gamma}_{N,\ell} - \Gamma_\ell), \ell = 1,\dots,L) \xrightarrow{\mathrm{d}} (\mathcal{G}_\ell, \ell = 1,\dots,L),$$

where $(\mathcal{G}_\ell, \ell = 1,\dots,L)$ is a zero-mean Gaussian random element with covariances (3.1).

(ii) If $d(s) \in (\frac{1}{4}, \frac{1}{2})$ for each $s \in \mathbb{S}$ and

$$\int_{\mathbb{S}} \frac{(\mathrm{E}(U\varepsilon(s))^4)^{\frac{1}{2}}}{2d(s)-1} \mu(ds) < \infty, \tag{3.11}$$

then,

$$(\Delta_N^{-1}(\widehat{\Gamma}_{N,\ell} - \Gamma_\ell), \ell = 1, \dots, L) \xrightarrow{\mathrm{d}} (Z, \ell = 1, \dots, L),$$

where Z is given in (3.8) and $\Delta_N^{-1} : \mathcal{S} \to \mathcal{S}$ with $\Delta_N^{-1}(\cdot) = N^{-H}(\cdot)N^{-H^*}$ and $H = T - \frac{1}{2}I$.

Remark 3.2. *It is worth comparing the assumptions (3.10) and (3.11) with the assumptions for the limit theorem for the sample mean of an $L^2(\mu)$-valued linear process in Proposition 4 in Characiejus and Račkauskas (2013). The integrals in (3.1) in Characiejus and Račkauskas (2013) given $d(s) \in (0, \frac{1}{4})$ can be bounded from above by the integral in (3.10). The integrals in (3.1) in Characiejus and Račkauskas (2013) given $d(s) \in (\frac{1}{4}, \frac{1}{2})$ can be bounded from above by the integral in (3.11). For this reason, whenever the assumptions (3.10) and (3.11) are satisfied, the asymptotic normality of the sample mean can be inferred as well.*

Example 3.3. *An example of a self-adjoint operator is the so-called convolution operator $F : L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \to L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ defined by*

$$F(f)(x) = \int_{\mathbb{R}} K(x-y)f(y)dy =: K * f(x)$$

with $K \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \cap L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) := \{f : \mathbb{R} \to \mathbb{R} | f$ measurable and $\int_{\mathbb{R}} |f(x)|^p dx < \infty\}$, $p = 1, 2$. The operator is self-adjoint if the kernel function K is Hermitian, i.e. if $K(-x) = \overline{K(x)}$. The Fourier transform defined by

$$(\mathcal{F}g)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t)e^{-ist}dt$$

with $s \in \mathbb{R}$ factorizes the convolution, that is, $\mathcal{F}(K * f) = \sqrt{2\pi}\mathcal{F}(K) \cdot \mathcal{F}(f)$. Defining the multiplication operator $D : L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \to L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda), f \mapsto d \cdot f$ by $d(s) = \sqrt{2\pi}(\mathcal{F}K)(s)$, we obtain the spectral decomposition in (2.1) of a convolution operator F, given by $F = \mathcal{F}^{-1}D\mathcal{F}$. See Section 3 in Düker (2018) for more details on this example.

We give an auxiliary result which is of independent interest. It states the asymptotic behavior of the sample autocovariances (2.4) of an $L^2(\mu)$-valued linear process as in (2.2). It will serve as a helpful tool to prove Theorem 3.1.

Lemma 3.4. *Let $\{X_n\}_{n \in \mathbb{Z}}$ be a stationary linear process (1.1) with values in $L^2(\mu)$ as in (2.2) and $\mathrm{E}\|\varepsilon_0\|^4_{L^2(\mu)} < \infty$.*

1. *If $d(s) \in (0, \frac{1}{4})$ for each $s \in \mathbb{S}$ and*

$$\int_{\mathbb{S}} \frac{(\mathrm{E}(\varepsilon(s))^4)^{\frac{1}{2}}}{d(s)^2} \mu(ds) < \infty, \tag{3.12}$$

then,

$$N^{\frac{1}{2}}(\widehat{\gamma}_{N,\ell}(r,s) - \gamma_\ell(r,s)), \ell = 0, \dots, L) \xrightarrow{\mathrm{d}} (G_\ell(r,s), \ell = 0, \dots, L),$$

where $(G_\ell, \ell = 0, \dots, L)$ is a zero-mean Gaussian random element with covariances $A_{I,\ell_1,\ell_2}(r_1, s_1, r_2, s_2)$ given in (3.2).

2. *If $d(s) \in (\frac{1}{4}, \frac{1}{2})$ and*

$$\int_{\mathbb{S}} \frac{(\mathrm{E}(\varepsilon(s))^4)^{\frac{1}{2}}}{2d(s)-1}\mu(ds) < \infty, \tag{3.13}$$

then,

$$N^{1-d(s)-d(r)}(\widehat{\gamma}_{N,\ell}(r,s) - \gamma_\ell(r,s), 0, \dots, L) \xrightarrow{\mathrm{d}} (\widetilde{Z}_I(r,s), \ell = 0, \dots, L),$$

where $\widetilde{Z}_I(r,s)$ is given in (3.8).

Remark 3.5. *Note that for r, s equal and fixed, the random variable $\widetilde{Z}_I(r,s)$ is a real-valued random element which coincides with the classical Rosenblatt process $Z_2(t)$ evaluated in one. See, for example Pipiras and Taqqu (2017), for more information about the Rosenblatt process.*

4 Double Wiener-Itô integral in function space

In this section, we introduce a double Wiener-Itô integral with values in $L^2(\nu \otimes \nu)$. We follow the analysis for real-valued Wiener-Itô integrals in Major (2014, Chapter 4).

Let $\{W^{(r)}(x) | x \in \mathbb{R}\}$ be an $L^2(\mu)$-valued standard Wiener process defined by the mapping $W : \mathbb{S} \to \mathbb{R}$, $r \mapsto \{W^{(r)}(x) | x \in \mathbb{R}\}$, where $W^{(r)}$ is a real-valued Gaussian process for fixed r. Then, the induced $L^2(\mu)$-valued Gaussian random measure satisfies

$$\begin{aligned}
&\mathrm{E}\, W^{(r)}(dx) = 0, \;\; \mathrm{E}\, W^{(r)}(dx) W^{(s)}(dx) = \sigma(r,s)dx, \;\; x \in \mathbb{R} \\
&\mathrm{E}\, W^{(r)}(dx) W^{(s)}(dy) = 0, \;\; x \neq y.
\end{aligned}$$

Let $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$ denote the space of all functions $f : \mathbb{R}^2 \to L^2(\nu \otimes \nu)$ equipped with the norm

$$\|f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \|f(x_1,x_2)\|^2_{L^2(\nu\otimes\nu)} dx_1 dx_2 < \infty.$$

In order to define a double Wiener-Itô integral in $L^2(\nu \otimes \nu)$, we first define the integral for simple functions and then use an approximation of f in terms of simple functions. The space of simple functions $S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$ is defined as follows. Partition the space $\mathbb{R}^2$ into cubes of size $\frac{1}{M}$ with $M \in \mathbb{N}$. Let $(\Delta) := \Delta_1 \times \Delta_2 \subset \mathbb{R}^2$ be such that $\Delta_1, \Delta_2 \in I_M$, where $I_M := \{(\frac{j}{M}, \frac{j+1}{M}], j \in \mathbb{Z}\}, M \in \mathbb{N}$. We write $(\Delta) \in \{\Delta_M^{\text{diag}}\}$, if $\Delta_1 = \Delta_2$ and $(\Delta) \in \{\Delta_M\}$, otherwise. The space $S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$ then consists of $L^2(\nu \otimes \nu)$-valued functions $f^{(r,s)}(x_1,x_2)$ on $\mathbb{R}^2$ satisfying

$$f^{(r,s)}(x_1,x_2) = \begin{cases} f^{(r,s)}_{\Delta_1\Delta_2}, & \text{if } (x_1,x_2) \in (\Delta), (\Delta) \in \{\Delta_M\}, \\ 0, & \text{if } (x_1,x_2) \in (\Delta), (\Delta) \in \{\Delta_M^{\text{diag}}\}, \end{cases}$$

where $f^{(r,s)}_{\Delta_1\Delta_2} \in L^2(\mu \otimes \mu)$.

For simple functions $f \in S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$, the double Wiener-Itô integral $I_2(f^{(r,s)})$ in $L^2(\nu \otimes \nu)$ is defined as

$$I_2(f^{(r,s)}) := \sum_{(\Delta)\in\{\Delta_M\}} f^{(r,s)}_{\Delta_1\Delta_2} W^{(r)}(\Delta_1) W^{(s)}(\Delta_2), \quad \text{with} \quad r, s \in \mathbb{S}. \tag{4.1}$$

The following lemma gives some properties regarding (4.1). Those properties will serve as basis to define double Wiener-Itô integrals with values in $L^2(\nu \otimes \nu)$.

Lemma 4.1. *For simple functions, the double Wiener-Itô integral (4.1) satisfies the following properties*

$$\mathrm{E}\, I_2(f) = 0, \tag{4.2}$$
$$\mathrm{E}\, \|I_2(f)\|^2_{L^2(\mu\otimes\mu)} \leqslant 2\|f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \tag{4.3}$$

for $f \in S_M(\mathbb{R}^2, L^2(\mu \otimes \mu))$.

Proof: The equality (4.2) follows since $(\Delta) \in \{\Delta_M\}$. The relation (4.3) holds since

$$\begin{aligned}
&\mathrm{E}\, \|I_2(f)\|^2_{L^2(\mu\otimes\mu)} \\
&= \mathrm{E} \int_{\mathbb{S}} \int_{\mathbb{S}} |I_2(f^{(r,s)})|^2 \mu(dr)\mu(ds) \\
&= \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{1}{M^2} \sum_{(\Delta)\in\{\Delta_M\}} (|f^{(r,s)}_{\Delta_1\Delta_2}|^2 \sigma^2(r)\sigma^2(s) + f^{(r,s)}_{\Delta_1\Delta_2} f^{(r,s)}_{\Delta_2\Delta_1} \sigma^2(r,s))\mu(dr)\mu(ds) \\
&\leqslant \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{1}{M^2} \sum_{(\Delta)\in\{\Delta_M\}} (|f^{(r,s)}_{\Delta_1\Delta_2}|^2 \sigma^2(r)\sigma^2(s) + |f^{(r,s)}_{\Delta_1\Delta_2} f^{(r,s)}_{\Delta_2\Delta_1}| \sigma^2(r)\sigma^2(s))\mu(dr)\mu(ds) \\
&\leqslant 2 \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{1}{M^2} \sum_{(\Delta)\in\{\Delta_M\}} |f^{(r,s)}_{\Delta_1\Delta_2}|^2 \sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds) \\
&\leqslant 2\|f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)}.
\end{aligned}$$

□

The relation (4.3) also proves that the integral (4.1) is a bounded linear operator from $\bigcup_{M=1}^{\infty} S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$ to $L^2_{L^2(\mu\otimes\mu)}(\Omega)$. The space $\bigcup_{M=1}^{\infty} S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$ is dense in $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$; see Grafakos (2008), p. 323–324. For this reason, for any $f \in L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$, there exists a sequence of simple functions $f_n \in S_{M_n}(\mathbb{R}^2, L^2(\nu \otimes \nu)), n \geqslant 1$ such that

$$\|f_n - f\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \to 0 \quad \text{as} \quad n \to \infty.$$

Then, $I_2(f_n)$ is a Cauchy sequence in $L^2_{L^2(\mu\otimes\mu)}(\Omega)$ since

$$\begin{aligned}
\mathrm{E}\, \|I_2(f_n) - I_2(f_m)\|^2_{L^2(\mu\otimes\mu)} &= \mathrm{E}\, \|I_2(f_n - f_m)\|^2_{L^2(\mu\otimes\mu)} \\
&\leqslant 2\|f_n - f_m\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \to 0 \quad \text{as} \quad n, m \to \infty.
\end{aligned}$$

It is left to prove that the limit does not depend on the approximating sequence f_n. Let $f_{1,n}$ and $f_{2,n}$ be two sequences in $S_{M_n}(\mathbb{R}^2, L^2(\nu \otimes \nu))$. Then,

$$\|f_{1,n} - f_{2,n}\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \leqslant \|f_{1,n} - f\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} + \|f - f_{2,n}\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \to 0.$$

Our analysis reveals that whenever we have a function $f \in L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$, there exists a sequence of simple functions $f_n \in S_{M_n}(\mathbb{R}^2, L^2(\nu \otimes \nu)), n \geqslant 1$, which converges to the function f in $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$. The sequence $I_2(f_n)$ converges in $L^2_{L^2(\mu\otimes\mu)}(\Omega)$ and the limit does not depend on the choice of the sequence f_n. This enables us to define double Wiener-Itô integrals in a function space.

Definition 4.2. *The double Wiener-Itô integral with kernel function f can be defined as*

$$I_2(f) = \int'_{\mathbb{R}^2} f^{(r,s)}(x_1, x_2) W^{(r)}(dx_1) W^{(s)}(dx_2) = \lim_{n\to\infty} I_2(f_n),$$

where f_n is a sequence of simple functions converging to f in the space $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$.

5 Proofs

In this section, we give the proof of our main result. Section 5.1 is concerned with the proof of Theorem 3.1. Section 5.2 gives the proof of Lemma 3.4.

5.1 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on Lemma 3.4. We show that a $\mathbb{H}$-valued linear process (1.1) with (1.2) can be written as a continuous function of an $L^2(\mu)$-valued linear process (1.1) with (1.4). Analogously, the sample autocovariance operator can be written as a continuous function of the sample autocovariances of an $L^2(\mu)$-valued linear process. Then, the continuous mapping theorem and Lemma 3.4 give the desired result.

Under the assumptions of Theorem 3.1, the underlying process $\{X_n\}_{n\in\mathbb{Z}}$ can be represented as a linear process (1.1) with (1.2). The spectral theorem for normal operators allows to write the normal operator T in (1.2) in terms of a multiplication operator D and a unitary operator U as in (2.1), that is,

$$\begin{aligned} X_n &= \sum_{j=0}^{\infty}(j+1)^{T-I}\varepsilon_{n-j} \\ &= U^*\left(\sum_{j=0}^{\infty}(j+1)^{D-I}(U\varepsilon_{n-j})\right) = U^* Z_n \end{aligned} \tag{5.1}$$

with

$$Z_n = \sum_{j=0}^{\infty}(j+1)^{D-I}(U\varepsilon_{n-j}), \tag{5.2}$$

where we used the fact that $e^T = \sum_{j=0}^{\infty} T^j/j!$ and $\lambda^T = e^{T\log(\lambda)}$ for $T \in L(\mathbb{H})$ and $\lambda > 0$.

The decomposition (5.1), enables us to prove some properties of $\{X_n\}_{n\in\mathbb{Z}}$. In Lemma 6.1, we give sufficient conditions for the series (5.2) to converge P-almost surely. The unitarity of U implies that $\{X_n\}_{n\in\mathbb{Z}}$ converges P-almost surely as well. To impose long-range dependence on the process $\{X_n\}_{n\in\mathbb{Z}}$, we supposed that the series of operator norms $\sum_{j=0}^{\infty}\|(j+1)^{T-I}\|_{op}$ diverges. Due to the unitarity of U, Lemma 6.2 in Section 6 implies that $\sum_{j=0}^{\infty}\|(j+1)^{T-I}\|_{op}$ diverges if and only if $\operatorname{ess\,sup}_{s\in\mathbb{S}} d(s) \geqslant 0$, where $d : \mathbb{S} \to \mathbb{R}$ defines the multiplication operator D in the representation (5.2).

The interchangeability of the series and the Hermitian adjoint of U in (5.1) is a consequence of the almost sure convergence of Z_n in (5.2) and the boundedness of the unitary operator U.

The process $\{Z_n\}_{n\in\mathbb{Z}}$ satisfies the assumptions in Lemma 3.4. The sequence $\{U\varepsilon_j\}_{j\in\mathbb{Z}}$ is an i.i.d. sequence with finite fourth moments since $U : \mathbb{H} \to L^2(\mu)$ is a unitary operator and $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is assumed to be an i.i.d. sequence with finite fourth moments.

The normalized sample autocovariance operator (1.3) of X_n can be written in terms of (5.1) as

$$\begin{aligned} &\Delta_N^{-1}(\widehat{\Gamma}_{N,\ell} - \Gamma_\ell) \\ &= N^{-T}\left(\frac{1}{N}\sum_{n=1}^{N}\langle X_{n+\ell}, N^{-T^*}(\cdot)\rangle X_n - E(\langle X_\ell, N^{-T^*}(\cdot)\rangle X_0)\right) \\ &= U^*\left(N^{\frac{1}{2}-d(s)}\int_{\mathbb{S}}(\widehat{\gamma}_{N,\ell}(r,s) - \gamma_{U,\ell}(r,s))N^{\frac{1}{2}-d(r)}U((\cdot)(r))\mu(dr)\right) \end{aligned} \tag{5.3}$$

with $\gamma_{U,\ell}$ as in (3.6). The relation (5.3) follows since

$$
\begin{aligned}
(\widehat{\Gamma}_{N,\ell} - \Gamma_\ell)(\cdot) &= \frac{1}{N}\sum_{n=1}^{N} \langle X_{n+\ell}, \cdot\rangle X_n - E(\langle X_\ell, \cdot\rangle X_0) \\
&= \frac{1}{N}\sum_{n=1}^{N} (\langle U^* Z_{n+\ell}, \cdot\rangle U^* Z_n - E(\langle U^* Z_\ell, \cdot\rangle U^* Z_0)) \\
&= U^*\left(\frac{1}{N}\sum_{k=1}^{N} (\langle Z_{n+\ell}, U(\cdot)\rangle_{L^2(\mu)} Z_n - E(\langle Z_\ell, U(\cdot)\rangle_{L^2(\mu)} Z_0)\right) \\
&= U^*\left(\int_{\mathbb{S}} \frac{1}{N}\sum_{n=1}^{N} (Z_{n+\ell}(r)Z_n(s) - \gamma_{U,\ell}(r,s))U((\cdot)(r))\mu(dr)\right),
\end{aligned}
$$

where we used the unitarity of U. The operator

$$
F : L^2(\mu\otimes\mu) \to \mathcal{S}, \quad A(r,s) \mapsto U^*\left(\int_{\mathbb{S}} A(r,s)U((\cdot)(r))\mu(dr)\right)
$$

is unitary. Lemma 6.3 and the unitarity of F also prove that the sample autocovariances of $\{X_n\}_{n\in\mathbb{Z}}$ belong P-almost surely to the space of Hilbert-Schmidt operators.

We can conclude that the continues mapping theorem, (5.3) and Lemma 3.4 give the desired convergence result.

5.2 Proof of Lemma 3.4

This section is concerned with the proof of Lemma 3.4. The section is divided into two parts concerning the cases $d(s) \in (0, \frac{1}{4})$ and $d(s) \in (\frac{1}{4}, \frac{1}{2})$ for each $s \in \mathbb{S}$, respectively.

Proof of part one of Lemma 3.4: The first part of Lemma 3.4 is concerned with the case $d(s) \in (0, \frac{1}{4})$ for each $s \in \mathbb{S}$. The proof is structured as follows. We will first present Lemma 5.1, which provides the resulting covariance structure of the limit and the required normalization sequence. It will also serve as a helpful tool for proving the sought convergence result. The final proof is stated after Lemma 5.1 and uses truncation techniques.

Lemma 5.1. *Under the assumptions of Lemma 3.4 with $d(s) \in (0, \frac{1}{4})$ for each $s \in \mathbb{S}$, there is a $C \in L^2(\mu\otimes\mu)\otimes L^2(\mu\otimes\mu)$, such that*

$$
\lim_{N\to\infty} N\,\mathrm{Cov}(\widehat{\gamma}_{N,\ell_1}, \widehat{\gamma}_{N,\ell_2}) = C
$$

for $\ell_1, \ell_2 \geqslant 0$, where the right hand side is defined in (3.2).

Proof: First, note that

$$
\mathrm{E}(\varepsilon_{j_1}(r_1)\varepsilon_{i_1}(s_1)\varepsilon_{i_2}(s_2)\varepsilon_{j_2}(r_2)) = \begin{cases}
\sigma^*(r_1,s_1,r_2,s_2), & \text{if } j_1 = i_1 = j_2 = i_2, \\
\sigma(r_1,s_1)\sigma(s_2,r_2), & \text{if } j_1 = i_1 \neq j_2 = i_2, \\
\sigma(r_1,r_2)\sigma(s_2,s_1), & \text{if } j_1 = j_2 \neq i_1 = i_2, \\
\sigma(r_1,s_2)\sigma(s_1,r_2), & \text{if } j_1 = i_2 \neq i_1 = j_2, \\
0, & \text{otherwise,}
\end{cases}
$$

where $\sigma(r,s)$ as in (2.3) and $\sigma^*(r_1,s_1,r_2,s_2) = \mathrm{E}(\varepsilon_0(r_1)\varepsilon_0(s_1)\varepsilon_0(s_2)\varepsilon_0(r_2))$. We are interested in the cross-autocovariances

$$\begin{aligned}
&\mathrm{Cov}(\widehat{\gamma}_{N,\ell_1}(r_1,s_1),\widehat{\gamma}_{N,\ell_2}(r_2,s_2))\\
&= \mathrm{E}((\widehat{\gamma}_{N,\ell_1}(r_1,s_1)-\gamma_{\ell_1}(r_1,s_1))(\widehat{\gamma}_{N,\ell_2}(r_2,s_2)-\gamma_{\ell_2}(r_2,s_2)))\\
&= \mathrm{E}(\widehat{\gamma}_{N,\ell_1}(r_1,s_1)\widehat{\gamma}_{N,\ell_2}(r_2,s_2)) - \gamma_{\ell_1}(r_1,s_1)\gamma_{\ell_2}(r_2,s_2)\\
&= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{l=1}^{N}\mathrm{E}(X_n(r_1)X_{n+\ell_1}(s_1)X_{l+\ell_2}(s_2)X_l(r_2)) - \gamma_{\ell_1}(r_1,s_1)\gamma_{\ell_2}(r_2,s_2).
\end{aligned}$$

We rewrite the expected value as

$$\begin{aligned}
&\mathrm{E}(X_n(r_1)X_{n+\ell_1}(s_1)X_{l+\ell_2}(s_2)X_l(r_2))\\
&= \mathrm{E}\Big(\sum_{i_1=0}^{\infty}(i_1+1)^{d(r_1)-1}\varepsilon_{n-i_1}(r_1)\sum_{j_1=0}^{\infty}(j_1+1)^{d(s_1)-1}\varepsilon_{n+\ell_1-j_1}(s_1)\\
&\qquad \sum_{j_2=0}^{\infty}(j_2+1)^{d(s_2)-1}\varepsilon_{l+\ell_2-j_2}(s_2)\sum_{i_2=0}^{\infty}(i_2+1)^{d(r_2)-1}\varepsilon_{l-i_2}(r_2)\Big)\\
&= \sum_{i_1,i_2,j_1,j_2=0}^{\infty}\mathrm{E}\Big(u_{i_1}(r_1)\varepsilon_{n-i_1}(r_1)u_{j_1+\ell_1}(s_1)\varepsilon_{n-j_1}(s_1)\\
&\qquad\qquad \times u_{j_2+\ell_2+l-n}(s_2)\varepsilon_{n-j_2}(s_2)u_{i_2+l-n}(r_2)\varepsilon_{n-i_2}(r_2)\Big)\\
&= \gamma_{\ell_1}(r_1,s_1)\gamma_{\ell_2}(r_2,s_2) + \gamma_{l-n}(r_1,r_2)\gamma_{\ell_2+l-n-\ell_1}(s_1,s_2) + \gamma_{\ell_2+l-n}(r_1,s_2)\gamma_{l-n-\ell_1}(s_1,r_2)\\
&\quad + \sum_{i=0}^{\infty}u_{i+1}(r_1)u_{i+\ell_1}(s_1)u_{i+l-n}(r_2)u_{i+\ell_2+l-n}(s_2)\Sigma_I(r_1,s_1,r_2,s_2)
\end{aligned}$$

with $u_j(s)$ as in (3.3) and $\Sigma_I(r_1,s_1,r_2,s_2)$ as in (3.4). Interchanging the order of summation yields

$$N\,\mathrm{Cov}(\widehat{\gamma}_{N,\ell_1}(r_1,s_1),\widehat{\gamma}_{N,\ell_2}(r_2,s_2)) = \sum_{|r|<N}\left(1-\frac{|r|}{N}\right)T_r((r_1,s_1),(r_2,s_2))$$

with

$$\begin{aligned}
&T_r((r_1,s_1),(r_2,s_2))\\
&= \gamma_r(r_1,r_2)\gamma_{\ell_2-\ell_1+r}(s_1,s_2) + \gamma_{\ell_2+r}(r_1,s_2)\gamma_{r-\ell_1}(s_1,r_2)\\
&\qquad + \sum_{i=0}^{\infty}u_i(r_1)u_{i+\ell_1}(s_1)u_{i+r}(r_2)u_{i+\ell_2+r}(s_2)\Sigma_I(r_1,s_1,r_2,s_2).
\end{aligned} \tag{5.4}$$

By Lemma 6.4 in Section 6 below, T_r is almost surely absolutely summable. For this reason,

$$\lim_{N\to\infty}N\,\mathrm{Cov}(\widehat{\gamma}_{N,\ell_1},\widehat{\gamma}_{N,\ell_2}) = T_r((r_1,s_1),(r_2,s_2)).$$

□

The first part of Lemma 3.4, can be proved by using Theorem 2 in Mas (2002). Therefore, we introduce the truncated versions of the quantities of interest

$$X_n^{(\kappa)}(r) = \sum_{j=0}^{\kappa}(j+1)^{d(r)-1}\varepsilon_{n-j}(r),\ \ \widehat{\gamma}_{N,\ell}^{(\kappa)}(r,s) = \frac{1}{N}\sum_{n=1}^{N}X_n^{(\kappa)}(r)X_{n+\ell}^{(\kappa)}(s)$$

and

$$\gamma_\ell^{(\kappa)}(r,s) = \mathrm{E}(X_0^{(\kappa)}(r)X_\ell^{(\kappa)}(s)) = \sum_{j=-\ell}^{\kappa-\ell} u_j(r)u_{j+\ell}(s)\sigma(r,s).$$

The truncated version $G_\ell^{(\kappa)}$ of the Gaussian limit G_ℓ can be defined as

$$\begin{aligned}
&\mathrm{E}(G_{\ell_1}^{(\kappa)}(r_1,s_1)G_{\ell_2}^{(\kappa)}(r_2,s_2)) \\
&= \sum_{r\in\mathbb{Z}}(\gamma_r(r_1,r_2)\gamma_{\ell_2-\ell_1+r}(s_1,s_2) + \gamma_{\ell_2+r}(r_1,s_2)\gamma_{r-\ell_1}(s_1,r_2)) \\
&\qquad + \sum_{r\in\mathbb{Z}}\sum_{i=0}^{\kappa} u_i(r_1)u_{i+\ell_1}(s_1)u_{i+r}(r_2)u_{i+\ell_2+r}(s_2)\Sigma_I(r_1,s_1,r_2,s_2).
\end{aligned}$$

We prove that

$$\lim_{N\to\infty} N^{\frac{1}{2}}((\widehat{\gamma}_{N,\ell}^{(\kappa)} - \gamma_\ell^{(\kappa)}), \ell = 0,\dots,L) \xrightarrow{\mathrm{d}} (G_\ell^{(\kappa)}, \ell = 0,\dots,L), \tag{5.5}$$

$$\lim_{\kappa\to\infty}(G_\ell^{(\kappa)}, \ell = 0,\dots,L) \xrightarrow{\mathrm{d}} (G_\ell, \ell = 0,\dots,L), \tag{5.6}$$

$$\lim_{\kappa\to\infty}\limsup_{N\to\infty} N \,\mathrm{Var}\, \|\widehat{\gamma}_{N,\ell}^{(\kappa)} - \widehat{\gamma}_{N,\ell}\|_{L^2(\mu\otimes\mu)} = 0. \tag{5.7}$$

The convergence (5.5) follows by Proposition 4 in Mas (2002), (5.6) is a consequence of

$$\lim_{\kappa\to\infty}\mathrm{E}(G_{\ell_1}^{(\kappa)}G_{\ell_2}^{(\kappa)}) = \mathrm{E}(G_{\ell_1}G_{\ell_2}),$$

since $\lim_{\kappa\to\infty}\gamma_\ell^{(\kappa)} = \gamma_\ell$, which requires $\sum_{j\in\mathbb{Z}}\|(j+1)^{D-I}\|_{op}^2 < \infty$. Condition (5.7) reduces to

$$\begin{aligned}
\lim_{\kappa\to\infty}\lim_{N\to\infty}\frac{1}{N}\mathrm{E}\Big(\sum_{n=1}^{N} X_n^{(\kappa)}(r)X_{n+\ell}^{(\kappa)}(s)\Big)^2 &= \mathrm{E}(G_\ell(r,s))^2, \\
\lim_{N\to\infty}\frac{1}{N}\mathrm{E}\Big(\sum_{n=1}^{N} X_n(r)X_{n+\ell}(s)\Big)^2 &= \mathrm{E}(G_\ell(r,s))^2, \\
\lim_{\kappa\to\infty}\lim_{N\to\infty}\frac{1}{N}\mathrm{E}\Big(\sum_{n=1}^{N} X_n^{(\kappa)}(r)X_{n+\ell}^{(\kappa)}(s)\sum_{n=1}^{N} X_n(r)X_{n+\ell}(s)\Big) &= \mathrm{E}(G_\ell(r,s))^2.
\end{aligned}$$

The convergence in $L^2(\mu\otimes\mu)$ then follows by Lemma 6.4.

Proof of part two of Lemma 3.4: This part of Lemma 3.4 is concerned with the case $d(s) \in (\frac{1}{4},\frac{1}{2})$. The proof is structured as follows. We will first separate the sample autocovariances into its diagonal and off-diagonal terms. Lemma 5.2 below will show that only the off-diagonal terms contribute to the limit in the sought convergence of the sample autocovariance operators. We will then provide a convergence result, Lemma 5.3 for second-order linear forms with respect to the components of an $L^2(\mu)$-valued i.i.d. process. Lemma 5.3 is the key result in our analysis and its application will give the sought convergence result.

The sample autocovariances can be separated into diagonal and off-diagonal parts

$$
\begin{aligned}
&\widehat{\gamma}_{N,\ell}(r,s) - \gamma_\ell(r,s) \\
&= \frac{1}{N}\sum_{n=1}^{N} X_n(r)X_{n+\ell}(s) - \sum_{j=0}^{\infty}(j+1)^{d(r)-1}(j+\ell+1)^{d(s)-1}\sigma(r,s) \\
&= \frac{1}{N}\sum_{n=1}^{N}\sum_{j=-\ell}^{\infty} u_j(r)u_{j+\ell}(s)(\varepsilon_{n-j}(r)\varepsilon_{n-j}(s) - \sigma(r,s)) \\
&\qquad + \frac{1}{N}\sum_{n=1}^{N}\sum_{j+\ell\neq i} u_j(r)u_i(s)\varepsilon_{n-j}(r)\varepsilon_{n-(i-\ell)}(s) \\
&= D_{N,\ell}(r,s) + O_{N,\ell}(r,s)
\end{aligned}
\tag{5.8}
$$

with $u_j(s)$ as in (3.3). The following Lemma 5.2 will imply that the convergence of the sample autocovariance operators can be proved only for the off-diagonal part.

Lemma 5.2. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be as in Lemma 3.4, with $d(s) \in (\frac{1}{4}, \frac{1}{2})$ for each $s \in \mathbb{S}$. Then,*

$$
N^{\frac{1}{2}}(D_{N,\ell}(r,s), \ell = 0, \ldots, L) \xrightarrow{\mathrm{d}} G(r,s), \tag{5.9}
$$

where $G(r,s)$ is a Gaussian random element with

$$
\begin{aligned}
&\mathrm{E}(G(r_1,s_1)G(r_2,s_2)) \\
&= \sum_{j\in\mathbb{Z}} u_j(r_1,s_1)(\sigma^*(r_1,s_1,r_2,s_2) - \sigma(r_1,s_1)\sigma(r_2,s_2))\sum_{j\in\mathbb{Z}} u_j(r_2,s_2)
\end{aligned}
$$

with $u_j(r,s) := u_j(r)u_{j+\ell}(s)$.

Proof: The convergence in (5.9) is a consequence of Theorem 2 in Merlevède et al. (1997). The diagonal part

$$
D_{N,\ell}(r,s) = \sum_{j=-\ell}^{\infty} u_j(r)u_{j+\ell}(s)(\varepsilon_{n-j}(r)\varepsilon_{n-j}(s) - \sigma(r,s))
$$

itself is an $L^2(\mu\otimes\mu)$-valued linear process with sequence of operators $u_j(r,s) := u_j(r)u_{j+\ell}(s)$ and i.i.d. sequence $\zeta_j(r,s) := \varepsilon_j(r)\varepsilon_j(s) - \sigma(r,s)$. That $D_{N,\ell}(r,s)$ belongs almost surely to $L^2(\mu\otimes\mu)$ follows under the assumptions (3.13) and $\mathrm{E}\,\|\varepsilon_j\|^4_{L^2(\mu)} < \infty$ by applying the same arguments as in Proposition 3 in Characiejus and Račkauskas (2013). Then, the assumptions of Theorem 2 in Merlevède et al. (1997) are satisfied since $u_j(r,s)$ is absolutely summable by the assumption $d(s) \in (\frac{1}{4}, \frac{1}{2})$ and $\{\zeta_j(r,s)\}_{j\in\mathbb{Z}}$ has finite second moments since $\mathrm{E}\,\|\varepsilon_j\|^4_{L^2(\mu)} < \infty$. □

To investigate the asymptotic behavior of the off-diagonal terms in (5.8), we further write

$$
\begin{aligned}
N^{1-d(r,s)}O_{N,\ell}(r,s) &= N^{-d(r,s)}\sum_{n=1}^{N}\sum_{j+\ell\neq i} u_j(r)u_i(s)\varepsilon_{n-j}(r)\varepsilon_{n-(i-\ell)}(s) \\
&= N^{-d(r,s)}\sum_{n=1}^{N}\sum_{j_1\neq j_2} u_{j_1}(r)u_{j_2+\ell}(s)\varepsilon_{n-j_1}(r)\varepsilon_{n-j_2}(s) \\
&= \sum_{j_1\neq j_2} N^{-d(r,s)}\sum_{n=1}^{N} u_{n-j_1}(r)u_{\ell+n-j_2}(s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s) \\
&= \sum_{j_1\neq j_2} C_{N,\ell}(j_1,j_2,r,s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s) \\
&=: Z_{N,\ell}(r,s),
\end{aligned}
$$

where

$$C_{N,\ell}(j_1, j_2, r, s) = N^{-d(r,s)} \sum_{n=1}^{N} u_{n-j_1}(r) u_{n+\ell-j_2}(s). \tag{5.10}$$

The following lemma provides a generalization of Proposition 14.3.2 in Giraitis et al. (2012).

Lemma 5.3. *Consider a linear combination of an off-diagonal tuple*

$$Q_2(C_N) = \sum_{j_1 \neq j_2} C_N(j_1, j_2, r, s) \varepsilon_{j_1}(r) \varepsilon_{j_2}(s).$$

Assume that the weights C_N are such that the functions

$$\widetilde{C}_N(x_1, x_2, r, s) = N C_N([x_1 N], [x_2 N], r, s), \quad x_1, x_2 \in \mathbb{R}$$

satisfy

$$\|\widetilde{C}_N - f\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \to 0$$

for a function $f \in L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$. Then $Q_2(C_N) \xrightarrow{\mathrm{d}} I_2(f)$ in $L^2(\mu \otimes \mu)$.

Proof: Let $f_\varepsilon^{(r,s)}$ be in $S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$ and define

$$C_{N,\varepsilon}(j_1, j_2, r, s) := N^{-1} f_\varepsilon^{(r,s)}\left(\frac{j_1}{N}, \frac{j_2}{N}\right), \quad j_1, j_2 \geqslant 0.$$

It is enough to prove that for all $\varepsilon > 0$, there exists $f_\varepsilon \in S_M(\mathbb{R}^2, L^2(\nu \otimes \nu))$, $M \geqslant 1$, such that

$$\operatorname{Var} \|Q_2(C_N) - Q_2(C_{N,\varepsilon})\|_{L^2(\mu\otimes\mu)} \leqslant \varepsilon, \tag{5.11}$$

$$\operatorname{Var} \|I_2(f_\varepsilon) - I_2(f)\|_{L^2(\mu\otimes\mu)} \leqslant \varepsilon, \tag{5.12}$$

$$Q_2(C_{N,\varepsilon}) \xrightarrow{f.d.d.} I_2(f_\varepsilon), \tag{5.13}$$

as $N \to \infty$. Note that

$$\mathrm{E}(\varepsilon_{i_1}(r_1)\varepsilon_{j_1}(s_1)\varepsilon_{i_2}(r_2)\varepsilon_{j_2}(s_2)) = \begin{cases} \sigma(r_1, r_2)\sigma(s_1, s_2), & \text{if } i_1 = i_2, j_1 = j_2, \\ \sigma(r_1, s_2)\sigma(s_1, r_2), & \text{if } i_1 = j_2, j_1 = i_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for (5.11)

$$\begin{aligned}
&\mathrm{E}\,\|Q_2(C_N)\|^2_{L^2(\mu\otimes\mu)} \\
&= \mathrm{E}\Big(\int_{\mathbb{S}}\int_{\mathbb{S}} |\sum_{j_1\neq j_2} C_N(j_1, j_2, r, s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)|^2 \mu(dr)\mu(ds)\Big) \\
&= \int_{\mathbb{S}}\int_{\mathbb{S}} \sum_{j_1\neq j_2}\sum_{i_2\neq i_2} C_N(j_1, j_2, r, s) C_N(i_1, i_2, r, s)\,\mathrm{E}(\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\varepsilon_{i_1}(r)\varepsilon_{i_2}(s))\mu(dr)\mu(ds) \\
&= \int_{\mathbb{S}}\int_{\mathbb{S}} \sum_{j_1\neq j_2} |C_N(j_1, j_2, r, s)|^2(\sigma^2(r)\sigma^2(s) + \sigma^2(r, s))\mu(dr)\mu(ds) \\
&\leqslant 2\int_{\mathbb{S}}\int_{\mathbb{S}}\int'_{\mathbb{R}^2} N^2|C_N([x_1N], [x_2N], r, s)|^2 dx_1 dx_2 \sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds) \\
&= 2\|\widetilde{C}_N\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)}.
\end{aligned}$$

This implies

$$\mathrm{E}\,\|Q_2(C_N) - Q_2(C_{N,\varepsilon})\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \leqslant 2\|\widetilde{C}_N - \widetilde{C}_{N,\varepsilon}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)},$$

where $\widetilde{C}_{N,\varepsilon}(x_1,x_2,r,s) = NC_{N,\varepsilon}([x_1N],[x_2N],r,s)$. The right hand side can be bounded by finding simple functions f_ε such that

$$\|\widetilde{C}_N - \widetilde{C}_{N,\varepsilon}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} < \varepsilon,$$

as $N \to \infty$. By assumption, there is a $N_0 \geqslant 1$ such that

$$\begin{aligned}
&\|\widetilde{C}_N - \widetilde{C}_{N_0}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \\
&\leqslant 2\|\widetilde{C}_N - f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} + 2\|f - \widetilde{C}_{N_0}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \leqslant \frac{\varepsilon}{6}
\end{aligned}$$

for all $N \geqslant N_0$. Given $N_0 \geqslant 1$ and $\varepsilon > 0$, there exist simple functions f_ε such that

$$\|\widetilde{C}_{N_0} - f_\varepsilon\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \leqslant \varepsilon/6.$$

The function $\widetilde{C}_{N,\varepsilon}$ derived from $C_{N,\varepsilon}$ satisfies

$$\|f_\varepsilon - \widetilde{C}_{N,\varepsilon}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \|f_\varepsilon(x_1,x_2) - f_\varepsilon\Big(\frac{\lfloor x_1N\rfloor}{N}, \frac{\lfloor x_2N\rfloor}{N}\Big)\|^2_{L^2(\nu\otimes\nu)} dx_1 dx_2 \to 0$$

as $N \to \infty$. Hence, there exists $\widetilde{N}_0 \geqslant 1$ such that

$$\begin{aligned}
&\|\widetilde{C}_N - \widetilde{C}_{N,\varepsilon}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \\
&\leqslant 3\|\widetilde{C}_N - \widetilde{C}_{N_0}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} + 3\|\widetilde{C}_{N_0} - \widetilde{C}_\varepsilon\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} + 3\|\widetilde{C}_\varepsilon - \widetilde{C}_{N,\varepsilon}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \\
&\leqslant \varepsilon/2.
\end{aligned}$$

This proves (5.12) since

$$\begin{aligned}
&\mathrm{Var}\,\|I(f_\varepsilon) - I(f)\|_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \\
&\leqslant 2\|f_\varepsilon - f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \\
&\leqslant 2(2\|f_\varepsilon - \widetilde{C}_{N_0}\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} + 2\|\widetilde{C}_{N_0} - f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)}) \leqslant \varepsilon.
\end{aligned}$$

Finally, for (5.13), note that

$$\begin{aligned}
Q_2(C_{N,\varepsilon}) &= \sum_{j_1\neq j_2} C_{N,\varepsilon}(j_1,j_2,r,s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s) \\
&= \sum_{j_1\neq j_2} N^{-1} f_\varepsilon^{(r,s)}\Big(\frac{j_1}{N},\frac{j_2}{N}\Big)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s) \\
&= \sum_{(\Delta)\in\{\Delta_M\}} f^{(r,s)}_{\varepsilon,\Delta_1\Delta_2} N^{-1} \sum_{j_1\neq j_2} \varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\mathbb{1}_{\{\frac{j_1}{N}\in\Delta_1,\frac{j_2}{N}\in\Delta_2\}} \\
&= \sum_{(\Delta)\in\{\Delta_M\}} f^{(r,s)}_{\varepsilon,\Delta_1\Delta_2} W_N^{(r)}(\Delta_1) W_N^{(s)}(\Delta_2),
\end{aligned}$$

where

$$W_N^{(r)}(\Delta_i) = N^{-\frac{1}{2}} \sum_{j:\frac{j}{N}\in\Delta_i} \varepsilon_j(r).$$

Since the intervals Δ_i are disjoint, $\{W^{(r)}(\Delta_i)\}_{i\in\mathbb{Z}}$ are independent random elements in $L^2(\mu)$. Since $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ are i.i.d. , the central limit theorem for Hilbert space-valued random elements applies and hence

$$(W_N^{(r)}(\Delta_{-J}), \dots, W_N^{(r)}(\Delta_J)) \xrightarrow{\mathrm{d}} (W^{(r)}(\Delta_{-J}), \dots, W^{(r)}(\Delta_J)).$$

Remember that $x \mapsto \langle x, A_\varepsilon x\rangle$ is a continuous mapping. Using the continuous mapping theorem yields

$$Q_2(C_{N,\varepsilon}) \xrightarrow{\mathrm{d}} \sum_{(\Delta)\in\{\Delta_M\}} f^{(r,s)}_{\varepsilon,\Delta_1\Delta_2} W^{(r)}(\Delta_1)W^{(s)}(\Delta_2) = I_2(f_\varepsilon).$$

□

To conclude the proof of Lemma 3.4, it is enough to prove that $C_{N,\ell}$ defined in (5.10) satisfies the assumptions of Lemma 5.3. Write

$$\begin{aligned}
&NC_{N,\ell}([x_1N],[x_2N],r,s)\\
&= NN^{1-d(r,s)} \sum_{n=1}^{N} u_{n-[x_1N]}(r)u_{n+\ell-[x_2N]}(s)\\
&= N^{2-d(r,s)} \int_0^1 u_{[vN]-[x_1N]}(r)u_{[vN]+\ell-[x_2N]}(s)dv,
\end{aligned}$$

then

$$\begin{aligned}
&N^{2-d(r,s)}u_{[vN]-[x_1N]}(r)u_{[vN]+\ell-[x_2N]}(s)\\
&= p_N^{(r)}(v,x_1)p_N^{(s)}(v,x_2)(v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1},
\end{aligned}$$

where

$$p_N^{(r)}(v,x) := \frac{N^{1-d(r)}u_{[vN]-[xN]}(r)}{(v-x)_+^{d(r)-1}} \to 1.$$

Furthermore, there are constants C_1, C_2, such that

$$\sup_{N\geqslant 1}\sup_{v,x_1} p_N^{(r)}(v,x_1) \leqslant C_1, \quad \sup_{N\geqslant 1}\sup_{v,x_2} p_N^{(s)}(v,x_2) \leqslant C_2,$$

which implies

$$\sup_{x_1,x_2} \Big|\int_0^1 N^{2-d(r,s)}u_{[vN]-[x_1N]}(r)u_{[vN]+\ell-[x_2N]}(s)dv\Big| \leqslant C\sup_{x_1,x_2}|f^{(r,s)}(x_1,x_2)|$$

and by the dominated convergence theorem

$$\int_0^1 N^{d(r)+d(s)}u_{[vN]-[x_1N]}(r)u_{[vN]+\ell-[x_2N]}(s)dv \to f^{(r,s)}(x_1,x_2),$$

where $f^{(r,s)}$ is defined in (3.9). Then, using Lemma 6.5, we can apply again the dominated convergence theorem. This leads to

$$\begin{aligned}
&\|\widetilde{C}_{N,\ell} - f\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)}\\
&\leqslant \Big\|\sum_{n=1}^{N} N^{d(r,s)-1}u_{n-[x_1N]}(r)u_{n+\ell-[x_2N]}(s) - f^{(r,s)}(x_1,x_2)\Big\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)} \to 0
\end{aligned}$$

This shows that the conditions in Lemma 5.3 are satisfied.

6 Some technical results and their proofs

In this section, we give some technical results and their proofs.

Lemma 6.1. *The series* (1.1) *with* u_j *as in* (1.4) *converges in mean square if and only if* $d(s) > \frac{1}{2}$ *for* μ*-almost all* $s \in \mathbb{S}$ *and if*

$$\int_{\mathbb{S}} \frac{\sigma^2(s)}{1-2d(s)} \mu(ds) < \infty.$$

Then, the series (1.1) *also converges almost surely.*

Proof: See Lemma A.1 in Düker (2018). □

Lemma 6.2. *The series of operator norms*

$$\sum_{j=0}^{\infty} \|(j+1)^{T-I}\|_{op}$$

diverges if and only if $\operatorname{ess\,sup}_{s\in\mathbb{S}} d(s) \geqslant 0$.

Proof: See p. 1445 in Düker (2018). □

Lemma 6.3. *The sample autocovariance operators* (2.4) *of* $\{X_n\}_{n\in\mathbb{Z}}$ *as in* (2.2) *belong* P*-almost surely to the space of Hilbert-Schmidt operators* $\mathcal{S}$*.*

Proof: The population quantity $\gamma_\ell(r,s)$ is indeed Hilbert-Schmidt; see p. 6 in Bosq (2000). For $\widehat{\gamma}_{N,\ell}(r,s)$, it is enough to prove that

$$\mathrm{E}\left(\int_{\mathbb{S}^2} |\widehat{\gamma}_{N,\ell}(r,s)|^2 \mu(dr)\mu(ds)\right) < \infty; \tag{6.1}$$

see p. 36 in Bosq (2000). For fixed N, the relation (6.1) is a consequence of

$$\begin{aligned}
&\mathrm{E}(X_n(r)X_{n+\ell}(s)X_{l+\ell}(s)X_l(r)) \\
&= \mathrm{E}\Big(\sum_{i_1=0}^{\infty}(i_1+1)^{d(r)-1}\varepsilon_{n-i_1}(r)\sum_{j_1=0}^{\infty}(j_1+1)^{d(s)-1}\varepsilon_{n+\ell-j_1}(s) \\
&\qquad \sum_{j_2=0}^{\infty}(j_2+1)^{d(s)-1}\varepsilon_{l+\ell-j_2}(s)\sum_{i_2=0}^{\infty}(i_2+1)^{d(r)-1}\varepsilon_{l-i_2}(r)\Big) \\
&= \sum_{i_1,i_2,j_1,j_2=0}^{\infty} \mathrm{E}\Big(u_{i_1}(r_1)\varepsilon_{n-i_1}(r_1)u_{j_1+\ell}(s)\varepsilon_{n-j_1}(s) \\
&\qquad\qquad \times u_{j_2+\ell+l-n}(s)\varepsilon_{n-j_2}(s)u_{i_2+l-n}(r)\varepsilon_{n-i_2}(r)\Big) \\
&= \gamma_\ell(r,s)\gamma_\ell(r,s) + \gamma_{l-n}(r,r)\gamma_{\ell+l-n-\ell}(s,s) + \gamma_{\ell+l-n}(r,s)\gamma_{l-n-\ell}(s,r) \\
&\quad + \sum_{i=0}^{\infty} u_{i+1}(r)u_{i+\ell}(s)u_{i+l-n}(r)u_{i+\ell+l-n}(s)\Sigma_I(r,s,r,s),
\end{aligned}$$

$\gamma_\ell(r,s)$ being Hilbert-Schmidt, and the assumptions (3.12) and (3.13). □

Lemma 6.4. *Under the assumptions of Lemma 3.4 with* $d(s) \in (0, \frac{1}{4})$ *for each* $s \in \mathbb{S}$*, the quantity* T_r *in* (5.4) *is almost surely absolutely summable.*

Proof: To prove the statement, it is necessary to ensure that

$$\int_{\mathbb{S}^4} \sum_{r\in\mathbb{Z}} |T_r((r_1,r_2),(s_1,s_2))|\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2) < \infty. \tag{6.2}$$

We prove the finiteness of the integral in (6.2) by considering the respective summands of T_r separately. For its first summand, note that

$$\begin{aligned}
&\int_{\mathbb{S}^4} \sum_{r\in\mathbb{Z}} |\gamma_r(r_1,r_2)\gamma_{\ell_2-\ell_1+r}(s_1,s_2)|\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2) \\
&\leqslant c\sum_{r\in\mathbb{Z}} \int_{\mathbb{S}^2} |r^{1-d(r_1,r_2)}|^2 \mu(dr_1)\mu(dr_2)\Big(\int_{\mathbb{S}} \frac{\sigma^2(s)}{d^2(s)}\mu(ds)\Big)^2 < \infty,
\end{aligned} \tag{6.3}$$

where $c > 0$ is a generic constant that may change from line to line. The relation (6.3) is a consequence of

$$\begin{aligned}
&\int_{\mathbb{S}^2} |\gamma_r(r_1,r_2)|\mu(dr_1)\mu(dr_2) \\
&= \int_{\mathbb{S}^2} |\sum_{j=0}^{\infty}(j+1)^{d(r_1)-1}(j+1+r)^{d(r_2)-1}\sigma(r_1,r_2)|\mu(dr_1)\mu(dr_2) \\
&\leqslant \int_{\mathbb{S}^2}\int_0^\infty x^{d(r_1)-1}(x+r)^{d(r_2)-1}dx\sigma(r_1,r_2)\mu(dr_1)\mu(dr_2) && (6.4)\\
&= \int_{\mathbb{S}^2} r^{d(r_1,r_2)-1}c(r_1,r_2)\sigma(r_1,r_2)\mu(dr_1)\mu(dr_2) \\
&\leqslant \int_{\mathbb{S}^2} r^{d(r_1,r_2)-1}\Big(\frac{1}{d(r_1)}+\frac{1}{1-d(r_1,r_2)}\Big)\sigma(r_1,r_2)\mu(dr_1)\mu(dr_2) && (6.5)\\
&\leqslant \int_{\mathbb{S}^2} r^{d(r_1,r_2)-1}\Big(\frac{1}{d(r_1)}+2\Big)(\sigma^2(r_1)\sigma^2(r_2))^{\frac{1}{2}}\mu(dr_1)\mu(dr_2) && (6.6)\\
&\leqslant 3\int_{\mathbb{S}^2} r^{d(r_1,r_2)-1}\frac{1}{d(r_1)}(\sigma^2(r_1)\sigma^2(r_2))^{\frac{1}{2}}\mu(dr_1)\mu(dr_2) && (6.7)\\
&\leqslant 3\Big(\int_{\mathbb{S}}\int_{\mathbb{S}} |r^{d(r_1,r_2)-1}|^2\mu(dr_1)\mu(dr_2)\int_{\mathbb{S}}\frac{\sigma^2(r)}{d^2(r)}\mu(dr)\int_{\mathbb{S}}\sigma^2(r)\mu(dr)\Big)^{\frac{1}{2}},
\end{aligned}$$

where we used the integral test for convergence of series in (6.4), Lemma 6.6 for (6.5), the inequality in (6.6) is a consequence of Lemma 6.6 and (6.7) follows since $d(r) \in (0, \frac{1}{4})$. For the last summand in (5.4), we consider each summand of $\Sigma_I(r_1,s_1,r_2,s_2)$ in (3.4) separately. We focus on the summand including $\sigma_I^*(r_1,s_1,r_2,s_2)$ since the other summands work analogously.

$$\begin{aligned}
&\int_{\mathbb{S}^4}\sum_{r\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}(i+1)^{d(r_1)-1}(i+\ell_1+1)^{d(s_1)-1}(i+r+1)^{d(r_2)-1}(i+\ell_2+r+1)^{d(s_2)-1}\\
&\qquad\times\sigma_I^*(r_1,s_1,r_2,s_2)\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2)\\
&\leqslant\int_{\mathbb{S}^4}\sum_{r\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}(i+1)^{d(r_1)-1}(i+1)^{d(s_1)-1}(i+r+1)^{d(r_2)-1}(i+r+1)^{d(s_2)-1}\\
&\qquad\times\sigma_I^*(r_1,s_1,r_2,s_2)\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2)\\
&\leqslant\int_{\mathbb{S}^4}\sum_{r\in\mathbb{Z}}\int_0^\infty x^{d(r_1)-1}x^{d(s_1)-1}(x+r)^{d(r_2)-1}(x+r)^{d(s_2)-1}dx\\
&\qquad\times\sigma_I^*(r_1,s_1,r_2,s_2)\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2)\\
&=\int_{\mathbb{S}^4}\sum_{r\in\mathbb{Z}}r^{d(r_1,s_1)+d(r_2,s_2)-3}c(r_1,s_1)c(r_2,s_2)\\
&\qquad\times\sigma_I^*(r_1,s_1,r_2,s_2)\mu(dr_1)\mu(ds_1)\mu(dr_2)\mu(ds_2)\\
&=\sum_{r\in\mathbb{Z}}r^{-2}\left(\int_{\mathbb{S}^2}c(r_1,r_2)(\sigma_I^*(r_1,r_1,r_2,r_2))^{\frac{1}{2}}\mu(dr_1)\mu(dr_2)\right)^2\\
&\leqslant\sum_{r\in\mathbb{Z}}r^{-2}\left(\int_{\mathbb{S}^2}\left(\frac{1}{d(r_1)}+\frac{1}{1-d(r_1,r_2)}\right)(\sigma_I^*(r_1,r_1,r_2,r_2))^{\frac{1}{2}}\mu(dr_1)\mu(dr_2)\right)^2\\
&\leqslant c\sum_{r\in\mathbb{Z}}r^{-2}\left(\int_{\mathbb{S}^2}\frac{1}{d(r_1)}(\sigma_I^*(r_1,r_1,r_2,r_2))^{\frac{1}{2}}\mu(dr_1)\mu(dr_2)\right)^2\\
&\leqslant c\sum_{r\in\mathbb{Z}}r^{-2}\int_{\mathbb{S}}\frac{(\sigma_I^*(r_1,r_1,r_1,r_1))^{\frac{1}{2}}}{d^2(r_1)}\mu(dr_1)\int_{\mathbb{S}}(\sigma_I^*(r_2,r_2,r_2,r_2))^{\frac{1}{2}}\mu(dr_2)
\end{aligned}$$

This expression is finite under assumption (3.13). The remaining summands of (5.4) work analogously. □

Lemma 6.5. *The function f in (3.9) takes values in $L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)$ if*

$$\int_{\mathbb{S}}\frac{\sigma^2(r)}{2d(r)-1}\mu(dr)<\infty \tag{6.8}$$

and $d(r)\in(\frac{1}{4},\frac{1}{2})$.

Proof: One can wirte

$$\begin{aligned}
&\|\int_0^1(v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\|^2_{L^2_{L^2(\nu\otimes\nu)}(\mathbb{R}^2)}\\
&\leqslant\frac{1}{2}\Big(\int_{\mathbb{S}}c(r)\sigma^2(r)\mu(dr)\Big)^2 &(6.9)\\
&\leqslant\frac{1}{2}\Big(\int_{\mathbb{S}}\Big(\frac{1}{1-d(r)}+\frac{1}{2d(r)-1}\Big)\sigma^2(r)\mu(dr)\Big)^2 &(6.10)\\
&\leqslant\frac{1}{2}\Big(2\int_{\mathbb{S}}\sigma^2(r)\mu(dr)+\int_{\mathbb{S}}\frac{\sigma^2(r)}{2d(r)-1}\mu(dr)\Big)^2,
\end{aligned}$$

where (6.10) follows by (6.13) and (6.9) by

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \Big| \int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1} dv \Big|^2 dx_1 dx_2 \\
&= \int_{\mathbb{R}^2} \int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1} dv \int_0^1 (u-x_1)_+^{d(r)-1}(u-x_2)_+^{d(s)-1} du dx_1 dx_2 \\
&= \int_0^1 \int_0^1 \int_{\mathbb{R}} (v-x_1)_+^{d(r)-1}(u-x_1)_+^{d(r)-1} dx_1 \int_{\mathbb{R}} (v-x_2)_+^{d(s)-1}(u-x_2)_+^{d(s)-1} dx_2 dv du \\
&= \int_0^1 \int_u^1 \int_{\mathbb{R}} (z_1)_+^{d(r)-1}(u-v+z_1)_+^{d(r)-1} dz_1 \\
&\qquad \times \int_{\mathbb{R}} (z_2)_+^{d(s)-1}(u-v+z_2)_+^{d(s)-1} dz_2 dv du \qquad (6.11) \\
&= \int_0^1 \int_u^1 \int_{\mathbb{R}} (x_1)_+^{d(r)-1}(1+x_1)_+^{d(r)-1} dx_1 \\
&\qquad \times \int_{\mathbb{R}} (x_2)_+^{d(s)-1}(1+x_2)_+^{d(s)-1} dx_2 (u-v)^{2d(r,s)-2} dv du \qquad (6.12) \\
&= B(1-d(r), 2d(r)-1) B(1-d(s), 2d(s)-1) \int_0^1 \int_u^1 (u-v)^{2d(r,s)-2} dv du \\
&= c(r)c(s) \frac{1}{(3-2d(r,s))(4-2d(r,s))} \\
&\leqslant c(r)c(s) \frac{1}{2},
\end{aligned}
$$

where $c(r) := c(r,r)$, for the definition of $c(r,s)$ see (2.5). The equation in (6.11) is received by substituting $x_1 = v - z_1$ and $x_2 = u - z_2$. Substituting further $z_1 = (u-v)x_1$ and $z_2 = (u-v)x_2$ gives (6.12) □

Lemma 6.6. *The function $c(r,s)$ in (2.5) can be bounded from above as*

$$c(r,s) \leqslant \frac{1}{d(r)} + \frac{1}{1-d(r,s)} \qquad (6.13)$$

for $d(r,s) = d(r) + d(s)$.

Proof: The Beta function in dependence of d can be written as

$$c(r,s) = \int_0^\infty x^{d(r)-1}(x+1)^{d(s)-1} dx = \int_0^1 x^{d(r)-1}(1-x)^{-d(r,s)} dx. \qquad (6.14)$$

The function in the second integral in (6.14) can be bounded from above by $x^{-d(r)} + (1-x)^{d(r,s)-2}$ for $x \in (0,1)$ since

$$x^{d(r)-1}(1-x)^{-d(r,s)} \leqslant x^{d(r)-1} + (1-x)^{-d(r,s)} \Leftrightarrow 1 \leqslant x^{1-d(r)} + (1-x)^{d(r,s)}. \qquad (6.15)$$

The inequality on the right hand side of the equivalence relation in (6.15) is satisfied, since for fixed $r, s \in \mathbb{S}$ the function $x^{1-d(r)} + (1-x)^{d(r,s)}$ has exactly one maximum and takes its minimum in its boundary points. The minimum is one. Then, (6.15) yields

$$c(r,s) \leqslant \int_0^1 x^{d(r)-1} dx + \int_0^1 (1-x)^{-d(r,s)} dx \leqslant \frac{1}{d(r)} + \frac{1}{1-d(r,s)}.$$

□

Acknowledgements: Research supported by the Research Training Group 2131 - *High-dimensional Phenomena in Probability - Fluctuations and Discontinuity.*

References

Aneiros, G., Cao, R., Fraiman, R., Genest, C., and Vieu, P. Recent advances in functional data analysis and high-dimensional statistics. *Journal of Multivariate Analysis*, 170:3–9, 2019.

Anh, V., Leonenko, N., Olenko, A., and Vaskovych, V. On rate of convergence in non-central limit theorems. *Bernoulli*, 25(4A):2920–2948, 2019.

Bai, S. and Taqqu, M. S. Behavior of the generalized Rosenblatt process at extreme critical exponent values. *Annals of Probability*, 45(2):1278–1324, 2017.

Bai, S. and Taqqu, M. S. Limit theorems for long-memory flows on Wiener chaos. *Bernoulli*, 26(2):1473–1503, 2020.

Beran, J., Feng, Y., Ghosh, S., and Kulik, R. *Long-Memory Processes: Probabilistic Properties and Statistical Methods*. Springer-Verlag Berlin Heidelberg, 2013.

Bosq, D. *Linear Processes in Function Spaces: Theory and Applications*. Lecture Notes in Statistics. Springer New York, 2000.

Characiejus, V. and Račkauskas, A. The central limit theorem for a sequence of random processes with space-varying long memory. *Lithuanian Mathematical Journal*, 53(2):149–160, 2013.

Characiejus, V. and Račkauskas, A. Operator self-similar processes and functional central limit theorems. *Stochastic Processes and their Applications*, 124(8):2605–2627, 2014.

Comway, J. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 1994.

Dette, H., Kokot, K., and Aue, A. Functional data analysis in the banach space of continuous functions. *Annals of Statistics*, 48(2):1168–1192, 2020.

Didier, G. and Pipiras, V. Integral representations and properties of operator fractional Brownian motions. *Bernoulli*, 17(1):1–33, 2011.

Düker, M.-C. Limit theorems for Hilbert space-valued linear processes under long-range dependence. *Stochastic Processes and their Applications*, 128(5):1439–1465, 2018.

Düker, M.-C. Limit theorems in the context of multivariate long-range dependence. *Stochastic Processes and their Applications*, 130(9):5394–5425, 2020.

Ferraty, F. and Vieu, P. *Nonparametric functional data analysis: theory and practice*. Springer Science & Business Media, 2006.

Fox, R. and Taqqu, M. S. Noncentral limit theorems for quadratic forms in random variables having long-range dependence. *Annals of Probability*, 13(2):428–446, 1985.

Giraitis, L., Koul, H., and Surgailis, D. *Large sample inference for long memory processes*. Imperial College Press, 2012.

Grafakos, L. *Classical Fourier Analysis*. Graduate Texts in Mathematics. Springer New York, 2008.

Horváth, L. and Kokoszka, P. Sample autocovariances of long-memory time series. *Bernoulli*, 14(2):405–418, 2008.

Horváth, L. and Kokoszka, P. *Inference for functional data with applications*, volume 200. Springer Science & Business Media, 2012.

Hörmann, S., Kokoszka, P., and Nisol, G. Testing for periodicity in functional time series. *Annals of Statistics*, 46(6A):2960–2984, 2018.

Jirak, M. Rate of convergence for Hilbert space valued processes. *Bernoulli*, 24(1):202–230, 2018.

Leonenko, N., Ruiz-Medina, M. D., and Taqqu, M. S. Non-central limit theorems for random fields subordinated to gamma-correlated random fields. *Bernoulli*, 23(4B):3469–3507, 2017.

Major, P. *Multiple Wiener-Itô Integrals: With Applications to Limit Theorems*. Springer International Publishing, 2014.

Mas, A. Weak convergence for the covariance operators of a Hilbertian linear process. *Stochastic Processes and their Applications*, 99(1):117–135, 2002.

Merlevède, F., Peligrad, M., and Utev, S. Sharp conditions for the CLT of linear processes in a Hilbert space. *Journal of Theoretical Probability*, 10(3):681–693, 1997.

Pipiras, V. and Taqqu, M. S. *Long-Range Dependence and Self-Similarity*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

Pipiras, V. and Taqqu, M. S. Central limit theorems for partial sums of bounded functionals of infinite-variance moving averages. *Bernoulli*, 9(5):833–855, 2003.

Ramsay, J. and Silverman, B. *Functional Data Analysis*. Springer New York, 1997.

Rosenblatt, M. Some limit theorems for partial sums of quadratic forms in stationary Gaussian variables. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 49(2):125–132, 1979.

Taqqu, M. S. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Probability Theory and Related Fields*, 31(4):287–302, 1975.

Taqqu, M. S. Convergence of integrated processes of arbitrary Hermite rank. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 50(1):53–83, 1979.

Tudor, C. A. Analysis of the Rosenblatt process. *ESAIM: Probability and Statistics*, 12:230–257, 2008.

van Delft, A. and Dette, H. A similarity measure for second order properties of non-stationary functional time series with applications to clustering and testing. *Bernoulli*, 27(1):469–501, 2021.

Veillette, M. S. and Taqqu, M. S. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli*, 19(3):982–1005, 2013.

Yan, L., Li, Y., and Wu, D. Approximating the Rosenblatt process by multiple Wiener integrals. *Electronic Communications in Probability*, 20(11):16 pp., 2015.