Supplementary material for "Testing for common structures in high-dimensional factor models" *†‡

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We present here some supplementary results for the article "Testing for common structures in high-dimensional factor models".

Recall from the article that the main idea is to transform a series which is assumed to follow a high-dimensional factor model and then apply an available change-point test. From a theoretical perspective, the main contribution is to show that the PCA estimators based on the transformed series are still consistent. The respective results are stated in Propositions B.1 and B.2 in the article including their proofs.

Those results on consistent estimation can then be used to infer the convergence of our test statistic as stated in Propositions 3.1–3.3. Inferring Propositions 3.1–3.3 from Propositions B.1 and B.2 follows arguments in Han and Inoue (2015). Therefore, we omitted those in the article. For completeness, we give the detailed proofs here as well as more details regarding the proofs under the alternative.

The result under the alternative hypothesis, stated in Proposition 4.1, can be proved following similar arguments. Appendix G supplements Section 4 and rephrases those arguments.

Recall also that our estimation procedure and our proofs for consistent estimation of the underlying factor model follow the presentation in Doz, Giannone, and Reichlin (2011). In contrast, Han and Inoue (2015) follow Bai (2003). While both lead to the same results, they differ in their assumptions. This document also has the purpose to clarify that the arguments in Han and Inoue (2015) go through by using the assumptions from Doz et al. (2011).

We see this supplementary material as a continuation of the actual article. We therefore start here with Appendix E following Appendix D in the main document. Accordingly, we adopt the notation of the article and refer to its labels.

E Proofs of results in Section 3.4

We state here the detailed proofs of our main results which are also rephrased in Appendix A.

Proof of Proposition 3.1: By Assumption W.1(ii), it suffices to prove that

$$|W(\widehat{F}) - W(FH_0)| = o_{\mathrm{p}}(1).$$

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For that, we follow the proof of Theorem 3(i) in Han and Inoue (2015). That is,

$$
\begin{aligned}
&\|W(\widehat{F}) - W(FH_0)\| \\
&\leq \left\| V(\widehat{F})'\left(\Omega^{-1}(\widehat{F}) - \Omega^{-1}(FH_0)\right)V(\widehat{F})\right\| + \left\| \left(V(\widehat{F}) - V(FH_0)\right)'\Omega^{-1}(FH_0)V(\widehat{F})\right\| \\
&\qquad + \left\| V(FH_0)'\Omega^{-1}(FH_0)\left(V(\widehat{F}) - V(FH_0)\right)\right\| \\
&\leq \left\| V(\widehat{F})\right\|^2 \left\|\Omega^{-1}(\widehat{F}) - \Omega^{-1}(FH_0)\right\| + \left\| V(\widehat{F}) - V(FH_0)\right\| \|\Omega^{-1}(FH_0)\| \left\| V(\widehat{F})\right\| \\
&\qquad + \|V(FH_0)\| \|\Omega^{-1}(FH_0)\| \left\| V(\widehat{F}) - V(FH_0)\right\| = o_{\mathrm{p}}(1). \qquad \text{(E.1)}
\end{aligned}
$$

In Q.1–Q.3 below we consider the different quantities in (E.1) separately to show the claimed asymptotic behavior. In addition, Propositions 3.2 and 3.3 are needed.

Q.1. Note that

$$
\begin{aligned}
\left\|\Omega^{-1}(\widehat{F}) - \Omega^{-1}(FH_0)\right\| &= \left\|\Omega^{-1}(\widehat{F})\left(\Omega(\widehat{F}) - \Omega(FH_0)\right)\Omega^{-1}(FH_0)\right\| \\
&\leq \|\Omega^{-1}(\widehat{F})\| \left\|\Omega(\widehat{F}) - \Omega(FH_0)\right\| \|\Omega^{-1}(FH_0)\| \\
&= \|\Omega^{-1}(\widehat{F})\| \|\Omega^{-1}(FH_0)\| o_{\mathrm{p}}(1) \qquad \text{(E.2)} \\
&= o_{\mathrm{p}}(1), \qquad \text{(E.3)}
\end{aligned}
$$

where (E.2) follows by Proposition 3.3 and (E.3) is due to Q.3 below.

Q.2. We get further that

$$
\left\| V(\widehat{F})\right\| \leq \left\| V(\widehat{F}) - V(FH_0)\right\| + \|V(FH_0)\| = o_{\mathrm{p}}(1) + \|V(FH_0)\| = \mathcal{O}_{\mathrm{p}}(1),
$$

where the first equality follows by Proposition 3.2. For the behavior of $\|V(FH_0)\|$,

$$
\begin{aligned}
\|V(FH_0)\| &= \left\| \operatorname{vech}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T/2} H_0' F_t F_t' H_0 - \frac{1}{\sqrt{T}} \sum_{t=T/2+1}^{T} H_0' F_t F_t' H_0 \right)\right\| \\
&\leq \sqrt{r} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T/2} F_t F_t' - \frac{1}{\sqrt{T}} \sum_{t=T/2+1}^{T} F_t F_t' \right\| \|H_0\|^2 = \mathcal{O}_{\mathrm{p}}(1)
\end{aligned}
$$

since $\|\operatorname{vech}(A)\| \leq \|\operatorname{vech}(A)\|_F \leq \|A\|_F \leq \sqrt{r}\|A\|$ for any matrix $A \in \mathbb{R}^{r\times r}$ and by Assumption A.2(i).

Q.3. We first consider $\left\|\Omega^{-1}(FH_0)\right\|$ and then $\left\|\Omega^{-1}(\widehat{F})\right\|$. Note that $\lambda_{\min}(\Omega) > 0$ since Ω in (3.22) is positive definite by Assumption W.1(i). Then,

$$
\begin{aligned}
\left\|\Omega^{-1}(FH_0)\right\| &= \lambda_{\max}\big(\Omega^{-1}(FH_0)\big) = \lambda_{\min}(\Omega(FH_0)) \\
&\leq \|\Omega(FH_0) - \Omega\| + \lambda_{\min}(\Omega) = o_{\mathrm{p}}(1) + \lambda_{\min}(\Omega), \qquad \text{(E.4)}
\end{aligned}
$$

where (E.4) is due to Weyl's Theorem (Theorem 4.3.1 in Horn and Johnson (2012)) and Assumption W.1(i).

Moving on to $\left\|\Omega^{-1}(\widehat{F})\right\|$, we first note that

$$\left\|\Omega(\widehat{F})-\Omega\right\| \leq \|\Omega(\widehat{F})-\Omega(FH_0)\|+\|\Omega(FH_0)-\Omega\| = o_{\mathrm{p}}(1)$$

by Proposition 3.3 and Assumption W.1(i). We can then show that

$$\begin{aligned}\left\|\Omega^{-1}(\widehat{F})\right\| &\leq |\lambda_{\min}(\Omega(\widehat{F}))-\lambda_{\min}(\Omega)|+\lambda_{\min}(\Omega)\\ &\leq \|\Omega(\widehat{F})-\Omega\|+\lambda_{\min}(\Omega) = o_{\mathrm{p}}(1)+\lambda_{\min}(\Omega).\end{aligned}$$

□

Proof of Proposition 3.2: We follow the arguments of the proof of Theorem 1 in Han and Inoue (2015). The proof requires our results (A.1) (Proposition B.1), (A.2) (Proposition B.2) and (A.3) (Lemma B.1). First, note that

$$\begin{aligned}\|V(\widehat{F})-V(FH_0)\| &= \|V(\widehat{F})-V(\widetilde{F}\widetilde{H})+V(\widetilde{F}\widetilde{H})-V(FH_0)\|\\ &\leq \|V(\widehat{F})-V(\widetilde{F}\widetilde{H})\|+\|V(\widetilde{F}\widetilde{H})-V(FH_0)\|.\end{aligned} \tag{E.5}$$

We consider the two summands in (E.5) separately and prove that they are both $o_{\mathrm{p}}(1)$.

For the first term in (E.5), we get

$$\begin{aligned}&\|V(\widehat{F})-V(\widetilde{F}\widetilde{H})\|\\ &= \frac{1}{\sqrt{T}}\left\|\operatorname{vech}\left(\sum_{t=1}^{T/2}\widehat{F}_t\widehat{F}_t'-\sum_{t=1}^{T/2}H_1'F_tF_t'H_1+\sum_{t=T/2+1}^{T}H_2'F_tF_t'H_2-\sum_{t=T/2+1}^{T}\widehat{F}_t\widehat{F}_t'\right)\right\|\\ &\leq \frac{1}{\sqrt{T}}\left\|\operatorname{vech}\left(\sum_{t=1}^{T/2}\widehat{F}_t\widehat{F}_t'-\sum_{t=1}^{T/2}H_1'F_tF_t'H_1\right)\right\|+\frac{1}{\sqrt{T}}\left\|\operatorname{vech}\left(\sum_{t=T/2+1}^{T}H_2'F_tF_t'H_2-\sum_{t=T/2+1}^{T}\widehat{F}_t\widehat{F}_t'\right)\right\|.\end{aligned} \tag{E.6}$$

The two summands in (E.6) can both be treated similarly. Therefore, we focus on the first summand. Using the same arguments as on page 36 in Han and Inoue (2014), we get

$$\begin{aligned}&\frac{1}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}\operatorname{vech}\Big(\widehat{F}_t\widehat{F}_t'-H_1'F_tF_t'H_1\Big)\right\|\\ &= \frac{1}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}\operatorname{vech}\Big(\widehat{F}_t(\widehat{F}_t'-F_t'H_1)+(\widehat{F}_t-H_1'F_t)F_t'H_1\Big)\right\|\\ &= \frac{1}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}\operatorname{vech}\Big((\widehat{F}_t-H_1'F_t)(\widehat{F}_t'-F_t'H_1)+H_1'F_t(\widehat{F}_t'-F_t'H_1)+(\widehat{F}_t-H_1'F_t)F_t'H_1\Big)\right\|\\ &\leq \frac{\sqrt{r}}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}(\widehat{F}_t-H_1'F_t)(\widehat{F}_t'-F_t'H_1)\right\|+\frac{2\sqrt{r}}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}H_1'F_t(\widehat{F}_t'-F_t'H_1)\right\|\\ &\leq \frac{\sqrt{r}}{\sqrt{T}}\sum_{t=1}^{T/2}\left\|\widehat{F}_t-H_1'F_t\right\|^2+\frac{2\sqrt{r}}{\sqrt{T}}\left\|\sum_{t=1}^{T/2}F_t(\widehat{F}_t'-F_t'H_1)\right\|\|H_1\|\end{aligned} \tag{E.7}$$

$$= \sqrt{T}\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^{2}}\right) = o_{\mathrm{p}}(1), \tag{E.8}$$

since by assumption $\sqrt{T}/d \to \infty$. The asymptotic (E.8) follows by Propositions B.1 and B.2 as well as Lemma B.1.

For the second summand in (E.5), we get

$$\begin{aligned}
&\|V(\widetilde{F}\widetilde{H}) - V(FH_0)\| \\
&= \frac{1}{\sqrt{T}}\left\|\mathrm{vech}\left(\sum_{t=1}^{T/2} H_1'F_tF_t'H_1 - \sum_{t=1}^{T/2} H_0'F_tF_t'H_0 + \sum_{t=T/2+1}^{T} H_0'F_tF_t'H_0 - \sum_{t=T/2+1}^{T} H_2'F_tF_t'H_2\right)\right\| \\
&\leq \frac{1}{\sqrt{T}}\left\|\sum_{t=1}^{T/2} \mathrm{vech}\big(H_1'F_tF_t'H_1 - H_0'F_tF_t'H_0\big)\right\| + \frac{1}{\sqrt{T}}\left\|\sum_{t=T/2+1}^{T} \mathrm{vech}\big(H_0'F_tF_t'H_0 - H_2'F_tF_t'H_2\big)\right\|.
\end{aligned} \tag{E.9}$$

Focusing on the first summand in (E.9), we can infer

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\left\|\sum_{t=1}^{T/2} \mathrm{vech}\big(H_1'F_tF_t'H_1 - H_0'F_tF_t'H_0\big)\right\| \\
&\leq \frac{\sqrt{r}}{\sqrt{T}}\sum_{t=1}^{T/2}\big\|H_1'F_t - H_0'F_t\big\|^2 + \frac{2\sqrt{r}}{\sqrt{T}}\left\|\sum_{t=1}^{T/2} F_t(F_t'H_1 - F_t'H_0)\right\|\|H_0\| && \text{(E.10)} \\
&\leq \frac{\sqrt{r}}{\sqrt{T}}\sum_{t=1}^{T/2}\|F_t\|^2\|H_1 - H_0\|^2 + \frac{2\sqrt{r}}{\sqrt{T}}\left\|\sum_{t=1}^{T/2} F_tF_t'\right\|\|H_1 - H_0\|\|H_0\| \\
&= \mathcal{O}_{\mathrm{p}}\big(\sqrt{T}\big)\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^{2}}\right) + \mathcal{O}_{\mathrm{p}}(1)\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}}\right) = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}}\right), && \text{(E.11)}
\end{aligned}$$

where (E.10) follows by the same arguments as (E.7) above. The asymptotics (E.11) then follow by Assumption A.2(i) and Lemma B.1. □

Proof of Proposition 3.3: The proof follows that of Theorem 2 in Han and Inoue (2015). Note that

$$\|\Omega(\widehat{F}) - \Omega(FH_0)\| \leq \|\Omega(\widehat{F}) - \Omega(\widetilde{F}\widetilde{H})\| + \|\Omega(\widetilde{F}\widetilde{H}) - \Omega(FH_0)\| = \mathcal{O}_{\mathrm{p}}\left(\frac{T^{\frac{1}{3}}}{\delta_{dT}}\right). \tag{E.12}$$

Consider the first summand in (E.12). We get

$$\begin{aligned}
&\|\Omega(\widehat{F}) - \Omega(\widetilde{F}\widetilde{H})\| \\
&\leq \left\|\widehat{\Gamma}_0(\widehat{F}) - \widehat{\Gamma}_0(\widetilde{F}\widetilde{H})\right\| \\
&\qquad + \left\|\sum_{j=1}^{T-1}\kappa\left(\frac{j}{b_T}\right)\left(\widehat{\Gamma}_j(\widehat{F}) + \widehat{\Gamma}_j'(\widehat{F})\right) - \sum_{j=1}^{T-1}\kappa\left(\frac{j}{b_T}\right)\left(\widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) + \widehat{\Gamma}_j'(\widetilde{F}\widetilde{H})\right)\right\| \\
&\leq \left\|\widehat{\Gamma}_0(\widehat{F}) - \widehat{\Gamma}_0(\widetilde{F}\widetilde{H})\right\| + 2\sum_{j=1}^{T-1}\left|\kappa\left(\frac{j}{b_T}\right)\right|\left\|\widehat{\Gamma}_j(\widehat{F}) - \widehat{\Gamma}_j(\widetilde{F}\widetilde{H})\right\|
\end{aligned}$$

$$\leq \left\| \widehat{\Gamma}_0(\widehat{F}) - \widehat{\Gamma}_0(\widetilde{F}\widetilde{H}) \right\| + 2\sum_{j=1}^{b_T} \left\| \widehat{\Gamma}_j(\widehat{F}) - \widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) \right\| = \mathcal{O}_{\mathrm{p}}\left(\frac{T^{\frac{1}{3}}}{\delta_{dT}} \right), \tag{E.13}$$

where (E.13) follows since the Bartlett kernel is supported on the interval $[-1, 1]$. The asymptotic behavior (E.13) then follows by (F.1) in Lemma F.1 and using Assumption W.2. We can infer further that the term is $o_{\mathrm{p}}(1)$ if $\frac{T^{\frac{2}{3}}}{d} \to 0$ as $d, T \to \infty$.

The result for the second summand in (E.12) can then be inferred by similar arguments but using (F.2) in Lemma F.1. □

F Auxiliary results and their proofs

We present here some auxiliary results used in the proofs of Propositions 3.1–3.3. Note that our results focus on the case when the change-point is known, i.e. when $\pi = 1/2$ in Han and Inoue (2014) and we consider only the long-run variance estimator based on the Bartlett kernel. The next lemma is analogous to Lemmas 7 and 8 in Han and Inoue (2014).

Lemma F.1. *Under Assumptions A.1–A.3, CR.1–CR.4,*

$$\sup_{0 \leq j \leq T-1} \left\| \widehat{\Gamma}_j(\widehat{F}) - \widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) \right\| = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}} \right) \tag{F.1}$$

and

$$\sup_{0 \leq j \leq T-1} \left\| \widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) - \widehat{\Gamma}_j(FH_0) \right\| = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}} \right). \tag{F.2}$$

Proof: Recall from (3.19) that

$$\widehat{\Gamma}_j(FH_0) = \frac{1}{T}\sum_{t=1+j}^{T} \operatorname{vech}(H_0' F_t F_t' H_0 - I_r) \operatorname{vech}(H_0' F_{t-j} F_{t-j}' H_0 - I_r)'.$$

We further note that

$$\begin{aligned}
\widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) = & \frac{1}{T}\sum_{t=1+j}^{T/2} \operatorname{vech}(H_1' F_t F_t' H_1 - I_r) \operatorname{vech}(H_1' F_{t-j} F_{t-j}' H_1 - I_r)' \\
& + \frac{1}{T}\sum_{t=1+j+T/2}^{T} \operatorname{vech}(H_2' F_t F_t' H_2 - I_r) \operatorname{vech}(H_2' F_{t-j} F_{t-j}' H_2 - I_r)' \\
& + \frac{1}{T}\sum_{t=T/2+1}^{T/2+j} \operatorname{vech}(H_2' F_t F_t' H_2 - I_r) \operatorname{vech}(H_1' F_{t-j} F_{t-j}' H_1 - I_r)' \\
=: & \widehat{\Gamma}_{1,j}(\widetilde{F}\widetilde{H}) + \widehat{\Gamma}_{2,j}(\widetilde{F}\widetilde{H}) + \widehat{\Gamma}_{3,j}(\widetilde{F}\widetilde{H}).
\end{aligned} \tag{F.3}$$

Similarly, split $\widehat{\Gamma}_j(\widehat{F})$ into $\widehat{\Gamma}_{1,j}(\widehat{F}) + \widehat{\Gamma}_{2,j}(\widehat{F}) + \widehat{\Gamma}_{3,j}(\widehat{F})$ according to the summations $\sum_{t=1+j}^{T/2}$, $\sum_{t=1+j+T/2}^{T}$ and $\sum_{t=T/2+1}^{T/2+j}$.

Proof of (F.1)*:* Using (F.3), we can write

$$\left\| \widehat{\Gamma}_j(\widehat{F}) - \widehat{\Gamma}_j(\widetilde{F}\widetilde{H}) \right\| \leq \left\| \widehat{\Gamma}_{1,j}(\widehat{F}) - \widehat{\Gamma}_{1,j}(\widetilde{F}\widetilde{H}) \right\| + \left\| \widehat{\Gamma}_{2,j}(\widehat{F}) - \widehat{\Gamma}_{2,j}(\widetilde{F}\widetilde{H}) \right\| + \left\| \widehat{\Gamma}_{3,j}(\widehat{F}) - \widehat{\Gamma}_{3,j}(\widetilde{F}\widetilde{H}) \right\|.$$

For the proofs, we focus on the first summand. The other two follow by similar considerations. Following the calculations on page 43 in Han and Inoue (2014), we get

$$
\begin{aligned}
&\left\|\widehat{\Gamma}_{1,j}(\widehat{F})-\widehat{\Gamma}_{1,j}(\widetilde{F}\widetilde{H})\right\| \\
&\leq \frac{\sqrt{r}}{T}\sum_{t=1+j}^{T/2}\|\widehat{F}_t\widehat{F}_t'\|\left\|\widehat{F}_{t-j}\widehat{F}_{t-j}'-H_1'F_{t-j}F_{t-j}'H_1\right\|+\frac{\sqrt{r}}{T}\sum_{t=1+j}^{T/2}\left\|\widehat{F}_t\widehat{F}_t'-H_1'F_tF_t'H_1\right\|\|H_1'F_{t-j}F_{t-j}'H_1\| \\
&\qquad+\frac{r}{T}\sum_{t=1+j}^{T/2}\left\|\widehat{F}_{t-j}\widehat{F}_{t-j}'-H_1'F_{t-j}F_{t-j}'H_1\right\|+\frac{r}{T}\sum_{t=1+j}^{T/2}\left\|\widehat{F}_t\widehat{F}_t'-H_1'F_tF_t'H_1\right\|. \qquad \text{(F.4)}
\end{aligned}
$$

All four summands in (F.4) can be treated similarly. Therefore, we focus on the first one and increase $T/2$ to T:

$$
\begin{aligned}
&\frac{1}{T}\sum_{t=1+j}^{T}\|\widehat{F}_t\widehat{F}_t'\|\left\|\widehat{F}_{t-j}\widehat{F}_{t-j}'-H_1'F_{t-j}F_{t-j}'H_1\right\| \\
&\leq \frac{1}{T}\sum_{t=1+j}^{T}\|\widehat{F}_t\widehat{F}_t'\|\left(\left\|\widehat{F}_{t-j}(\widehat{F}_{t-j}'-F_{t-j}'H_1)\right\|+\left\|(\widehat{F}_{t-j}-H_1'F_{t-j})F_{t-j}'H_1\right\|\right) \\
&\leq \left(\frac{1}{T}\sum_{t=1+j}^{T}\|\widehat{F}_t\|^4\right)^{\frac{1}{2}}\left(\left(\frac{1}{T}\sum_{t=1+j}^{T}\|\widehat{F}_{t-j}\|^4\frac{1}{T}\sum_{t=1}^{T}\|\widehat{F}_t'-F_t'H_1\|^4\right)^{\frac{1}{4}}\right. \\
&\qquad\left.+\left(\frac{1}{T}\sum_{t=1}^{T}\|\widehat{F}_t'-F_t'H_1\|^4\frac{1}{T}\sum_{t=1+j}^{T}\|F_{t-j}\|^4\|H_1\|^4\right)^{\frac{1}{4}}\right)=\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}}\right), \qquad \text{(F.5)}
\end{aligned}
$$

where (F.5) is due to Lemmas F.2 and B.1.

Proof of (F.2)*:* Using (F.3), we can write

$$
\begin{aligned}
\left\|\widehat{\Gamma}_j(\widetilde{F}\widetilde{H})-\widehat{\Gamma}_j(FH_0)\right\| \leq &\left\|\widehat{\Gamma}_{1,j}(\widetilde{F}\widetilde{H})-\widehat{\Gamma}_{1,j}(FH_0)\right\|+\left\|\widehat{\Gamma}_{2,j}(\widetilde{F}\widetilde{H})-\widehat{\Gamma}_{2,j}(FH_0)\right\| \\
&+\left\|\widehat{\Gamma}_{3,j}(\widetilde{F}\widetilde{H})-\widehat{\Gamma}_{3,j}(FH_0)\right\|.
\end{aligned}
$$

We focus on the first summand. The other two can be dealt with similarly. As in (F.5), we get

$$
\begin{aligned}
&\left\|\widehat{\Gamma}_{1,j}(\widetilde{F}\widetilde{H})-\widehat{\Gamma}_{1,j}(FH_0)\right\| \\
&\leq \frac{\sqrt{r}}{T}\sum_{t=j+1}^{T/2}\left\|H_1'F_tF_t'H_1\right\|\left\|H_1'F_{t-j}F_{t-j}'H_1-H_0'F_{t-j}F_{t-j}'H_0\right\| \\
&\qquad+\frac{\sqrt{r}}{T}\sum_{t=j+1}^{T/2}\left\|H_1'F_tF_t'H_1-H_0'F_tF_t'H_0\right\|\left\|H_0'F_{t-j}F_{t-j}'H_0\right\|. \\
&\qquad+\frac{r}{T}\sum_{t=j+1}^{T/2}\left\|H_1'F_{t-j}F_{t-j}'H_1-H_0'F_{t-j}F_{t-j}'H_0\right\|+\frac{r}{T}\sum_{t=j+1}^{T/2}\left\|H_1'F_tF_t'H_1-H_0'F_tF_t'H_0\right\| \qquad \text{(F.6)}
\end{aligned}
$$

All four summands in (F.6) can be treated similarly. For the first one, we get

$$
\frac{1}{T}\sum_{t=j+1}^{T/2}\left\|H_1'F_tF_t'H_1\right\|\left\|H_1'F_{t-j}F_{t-j}'H_1-H_0'F_{t-j}F_{t-j}'H_0\right\|
$$

$$\leq \left(\frac{1}{T}\sum_{t=1+j}^{T/2}\|F_t'H_1\|^4\right)^{\frac{1}{2}}\left(\frac{1}{T}\sum_{t=1+j}^{T/2}\|H_1'F_{t-j}F_{t-j}'(H_1-H_0)+(H_1-H_0)'F_{t-j}F_{t-j}'H_0\|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{T}\sum_{t=1+j}^{T/2}\|F_t'H_1\|^4\right)^{\frac{1}{2}}\left((\|H_1\|^2+\|H_0\|^2)\frac{1}{T}\sum_{t=1}^{T}\|F_t\|^4\right)^{\frac{1}{2}}\|H_1-H_0\| = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}}\right), \tag{F.7}$$

where (F.7) is due to Lemmas F.2 and B.1. □

The following lemma is analogous to Lemma 5 in Han and Inoue (2014).

Lemma F.2. *Under Assumptions A.1–A.3, CR.1–CR.4,*

$$\frac{1}{T}\sum_{t=1}^{T/2}\|\widehat{F}_t-H_1'F_t\|^4 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right),\ \ \frac{1}{T}\sum_{t=T/2+1}^{T}\|\widehat{F}_t-H_2'F_t\|^4 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right) \tag{F.8}$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\|\widehat{F}_t\|^4 = \mathcal{O}_{\mathrm{p}}(1). \tag{F.9}$$

Proof: We first prove (F.8) and then infer (F.9).

Proof of (F.8)*:* Recall from (B.10) the following relationship

$$\widehat{F}-\widetilde{F}\widetilde{H} = \frac{1}{dT}(YY'-\widetilde{F}\Lambda_P\widetilde{F}')(\widehat{F}-F)\widehat{V}_r^{-1} + \frac{1}{dT}(YY'-\widetilde{F}\Lambda_P\widetilde{F}')F\widehat{V}_r^{-1}.$$

Then, with e_t denoting the tth unit vector in $\mathbb{R}^T$,

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T/2}\|\widehat{F}_t-H_1'F_t\|^4 &= \frac{1}{T}\sum_{t=1}^{T/2}\|e_t'(\widehat{F}-\widetilde{F}\widetilde{H})\|^4 \\
&\leq c\frac{1}{T}\sum_{t=1}^{T/2}\|\frac{1}{dT}e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)(\widehat{F}-F)\widehat{V}_r^{-1}\|^4 \\
&\quad + c\frac{1}{T}\sum_{t=1}^{T/2}\|\frac{1}{dT}e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)F\widehat{V}_r^{-1}\|^4 \\
&\leq c\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)\|^4\frac{1}{T^2}\|\widehat{F}-F\|^4\|\widehat{V}_r^{-1}\|^4 \\
&\quad + c\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\|e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)F\|^4\|\widehat{V}_r^{-1}\|^4 \\
&\leq c\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)\|^4\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right)\mathcal{O}_{\mathrm{p}}(1) \\
&\quad + c\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\|e_t'\Big(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\Big)F\|^4\mathcal{O}_{\mathrm{p}}(1),
\end{aligned} \tag{F.10}$$

where (F.10) is due to Proposition C.1 and Corollary C.1*(ii)* (together with Assumption CR.1). For the two remaining quantities in (F.10), we get, using (B.11) and the fact that e_t is considered with $t = 1, \dots, T/2$,

$$\begin{aligned}
&\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'\Big(YY' - \widetilde{F}\Lambda_P\widetilde{F}'\Big)\|^4 \\
&\leq c\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\big(\|e_t'A\|^4 + \|e_t'C\|^4\big) = \mathcal{O}_{\mathrm{p}}(1), \qquad \text{(F.11)}
\end{aligned}$$

with e_t now being viewed as the tth unit vector in $\mathbb{R}^{T/2}$. We focus on the second summand in (F.11); the first one can be treated similarly. We have

$$\|e_t'C\|^4 = \|e_t'F^b\Lambda_2'\widehat{P}_I^{1\prime}\widehat{P}_I^2\varepsilon^{a\prime} + e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2F^{a\prime} + e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\varepsilon^{a\prime}\|^4. \qquad \text{(F.12)}$$

For the summands in (F.12), we get

$$\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'F^b\Lambda_2'\widehat{P}_I^{1\prime}\widehat{P}_I^2\varepsilon^{a\prime}\|^4 &= \frac{1}{T}\sum_{t=1}^{T/2}\|e_t'F\|^4\frac{1}{d^4T^2}\|\Lambda_2'\widehat{P}_I^{1\prime}\widehat{P}_I^2\varepsilon^{a\prime}\|^4 \\
&= \frac{1}{T}\sum_{t=1}^{T/2}\|e_t'F\|^4\frac{1}{d^4T^2}\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right) \qquad \text{(F.13)} \\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right) = \mathcal{O}_{\mathrm{p}}(1), \qquad \text{(F.14)}
\end{aligned}$$

where (F.13) can be proved similarly to (B.25) and (F.14) follows by R.1 below.

For the second summand in (F.12), note that

$$\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2F^{a\prime}\|^4 &\leq \frac{1}{d^4T}\sum_{t=1}^{T/2}\|e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2\|^4\frac{1}{T^2}\|F\|^4 \\
&\leq \frac{1}{d^4T}\sum_{t=1}^{T/2}\|e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2\|^4\mathcal{O}_{\mathrm{p}}(1) \qquad \text{(F.15)} \\
&\leq \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right) = \mathcal{O}_{\mathrm{p}}(1), \qquad \text{(F.16)}
\end{aligned}$$

where (F.15) is due to S.1 in the proof of Proposition B.1 and (F.16) follows since

$$\begin{aligned}
\frac{1}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2\|^4 &\leq \frac{8}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b\Lambda_2\|^4 + \frac{8}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b(\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2 - \Lambda_2)\|^4 \\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d^2}\right) + \frac{8}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b\|^4\|\widehat{P}_I^{1\prime}\widehat{P}_I^2\Lambda_2 - \Lambda_2\|^4 \qquad \text{(F.17)} \\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d^2}\right) + \frac{1}{d^4T}\mathcal{O}_{\mathrm{p}}(d^2T)d^2\left(\mathcal{O}_{\mathrm{p}}\left(\frac{1}{d^4}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{T^2}\right)\right) \qquad \text{(F.18)} \\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right), \qquad \text{(F.19)}
\end{aligned}$$

where (F.17) is due to Assumption A.3(ii), (F.18) follows from R.2 below and a bound similar to (B.47) with the first factor d replaced by $\sqrt{d}$.

For the third summand in (F.12), we get

$$
\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^2}\sum_{t=1}^{T/2}\|e_t'\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^2\varepsilon^{a'}\|^4 &\leq \frac{1}{Td^2}\sum_{t=1}^{T/2}\|e_t'\varepsilon\|^4\frac{1}{d^2T^3}\|\varepsilon\|^4\|\widehat{P}_I^{1'}\widehat{P}_I^2\|^4 \\
&\leq \mathcal{O}_{\mathrm{p}}(1)\frac{1}{d^2T^3}\left(\mathcal{O}_{\mathrm{p}}\left(d\sqrt{T}\right)+\mathcal{O}(T)\right)^2 && \text{(F.20)}\\
&\leq \mathcal{O}_{\mathrm{p}}(1), && \text{(F.21)}
\end{aligned}
$$

where (F.20) is due to S.2 in the proof of Proposition B.1 and (F.21) follows by R.2.

For the second summand in (F.10), by using (B.11),

$$
\begin{aligned}
&\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\|e_t'\left(YY'-\widetilde{F}\Lambda_P\widetilde{F}'\right)F\|^4 \\
&\leq \frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\left\|e_t'\begin{pmatrix}AF^b+CF^a\\ C'F^b+BF^a\end{pmatrix}\right\|^4 \\
&\leq \frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\|e_t'AF^b+e_t'CF^a\|^4 \\
&\leq c\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T/2}\left(\|e_t'AF^b\|^4+\|e_t'CF^a\|^4\right)=\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right). && \text{(F.22)}
\end{aligned}
$$

We consider only one of the summands in (F.22). Recall from (B.23) that

$$
AF^b=F^b\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\varepsilon^{b'}F^b+\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2F^{b'}F^b+\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^1\varepsilon^{b'}F^b. \tag{F.23}
$$

We consider the three summands in (F.23) separately. For the first one,

$$
\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T}\|e_t'F^b\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\varepsilon^{b'}F^b\|^4 &\leq \frac{1}{T}\sum_{t=1}^{T}\|e_t'F\|^4\|\frac{1}{d^4T^4}\|\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\varepsilon^{b'}F^b\|^4 \\
&=\frac{1}{T}\sum_{t=1}^{T}\|e_t'F\|^4\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^8}\right) && \text{(F.24)}\\
&=\mathcal{O}_{\mathrm{p}}(1)\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^8}\right)=\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right), && \text{(F.25)}
\end{aligned}
$$

where (F.24) follows by (B.39) and (F.25) is due to R.1 below.

For the second summand in (F.23), note that

$$
\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T}\|e_t'\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2F^{b'}F^b\|^4 &\leq \frac{1}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2\|^4\frac{1}{T^4}\|F\|^8 \\
&=\frac{1}{d^4T}\sum_{t=1}^{T}\|e_t'\varepsilon^b\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2\|^4\mathcal{O}_{\mathrm{p}}(1) && \text{(F.26)}
\end{aligned}
$$

$$= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d^2}\right) = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right), \tag{F.27}$$

where (F.26) is due to S.1 in the proof of Proposition B.1 and (F.27) follows similarly to (F.19) above.

Finally, the third summand in (F.23) can be bounded as

$$\begin{aligned}
\frac{1}{T}\frac{1}{d^4T^4}\sum_{t=1}^{T}\|e_t'\varepsilon^b\widehat{P}_I^{1\prime}\widehat{P}_I^1\varepsilon^{b\prime}F^b\|^4 &\leq \frac{1}{d^2T^3}\sum_{t=1}^{T}\|e_t'\varepsilon\|^4\frac{1}{d^2T^2}\|\varepsilon^{b\prime}F^b\|^4\|\widehat{P}_I^{1\prime}\widehat{P}_I^1\|^4 \\
&\leq \frac{1}{d^2T^3}\sum_{t=1}^{T}\|e_t'\varepsilon\|^4\mathcal{O}_{\mathrm{p}}(1) \qquad \text{(F.28)} \\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{T^2}\right) = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right), \qquad \text{(F.29)}
\end{aligned}$$

where the asymptotics of $(\varepsilon^{b\prime}F^b)/T$ in (F.28) follow by the same calculations as done on p. 199 in Doz et al. (2011) as part of the proof of Lemma 2(i) under Assumptions A.2 and CR.3. Furthermore, (F.29) follows by R.2 below under Assumption A.3(iii).

R.1. Under Assumption A.2, $\frac{1}{T}\sum_{t=1}^{T}\|e_t'F\|^4 = \mathcal{O}_{\mathrm{p}}(1)$ since

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\,\|e_t'F\|^4 \\
&= \frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left\|\sum_{k=0}^{\infty}C_k a_{t-k}\right\|^4 \\
&\leq \frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left(\sum_{k=0}^{\infty}\|C_k\|\|a_{t-k}\|\right)^4 \\
&\leq \frac{1}{T}\sum_{t=1}^{T}\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\|C_{k_1}\|\|C_{k_2}\|\|C_{k_3}\|\|C_{k_4}\|\,\mathrm{E}(\|a_{t-k_1}\|\|a_{t-k_2}\|\|a_{t-k_3}\|\|a_{t-k_4}\|) \\
&\leq \frac{1}{T}\sum_{t=1}^{T}\sum_{k_1,k_2,k_3,k_4=0}^{\infty}\|C_{k_1}\|\|C_{k_2}\|\|C_{k_3}\|\|C_{k_4}\|\big(\mathrm{E}\,\|a_{t-k_1}\|^4\,\mathrm{E}\,\|a_{t-k_2}\|^4\,\mathrm{E}\,\|a_{t-k_3}\|^4\,\mathrm{E}\,\|a_{t-k_4}\|^4\big)^{\frac{1}{4}} \\
&\leq c\left(\sum_{k=0}^{\infty}\|C_k\|\right)^4 \qquad \text{(F.30)}
\end{aligned}$$

for some constant c, where (F.30) follows using Cauchy-Schwarz inequality

$$\begin{aligned}
\mathrm{E}\,\|a_{t-k}\|^4 = \mathrm{E}\left(\sum_{i=1}^{r}|a_{i,t-k}|^2\right)^2 &= \sum_{i_1,i_2=1}^{r}\mathrm{E}(|a_{i_1,t-k}|^2|a_{i_2,t-k}|^2) \\
&\leq \sum_{i_1,i_2=1}^{r}(\mathrm{E}(|a_{i_1,t-k}|^4)\,\mathrm{E}(|a_{i_2,t-k}|^4))^{\frac{1}{2}} < \infty
\end{aligned}$$

by Assumption A.2(i) and stationarity of $\{a_t\}$.

R.2. Under Assumption A.3(iii), it holds that $\sum_{t=1}^{T}\|e_t'\varepsilon\|^4 = \mathcal{O}_{\mathrm{p}}(d^2T)$ since

$$\begin{aligned}
\sum_{t=1}^{T} \mathrm{E}\,\|e_t'\varepsilon\|^4 &= \mathrm{E}\sum_{t=1}^{T}\left(\sum_{u=1}^{d}\varepsilon_{t,u}^2\right)^2 \\
&= \sum_{t=1}^{T}\sum_{u_1,u_2=1}^{d} \mathrm{E}(\varepsilon_{t,u_1}^2\varepsilon_{t,u_2}^2) \\
&\le \sum_{t=1}^{T}\sum_{u_1,u_2=1}^{d} (\mathrm{E}(\varepsilon_{t,u_1}^4)\,\mathrm{E}(\varepsilon_{t,u_2}^4))^{\frac{1}{2}} \qquad (\text{F.31})\\
&\le d^2TM, \qquad (\text{F.32})
\end{aligned}$$

where (F.31) uses Cauchy-Schwarz inequality and (F.32) is due to Assumption A.3(iii).

Proof of (F.9)*:* Note that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\|\widehat{F}_t\|^4 &= \frac{1}{T}\sum_{t=1}^{T/2}\|\widehat{F}_t\|^4 + \frac{1}{T}\sum_{t=T/2+1}^{T}\|\widehat{F}_t\|^4 \\
&= \frac{1}{T}\sum_{t=1}^{T/2}\|\widehat{F}_t - H_1'F_t + H_1'F_t\|^4 + \frac{1}{T}\sum_{t=T/2+1}^{T}\|\widehat{F}_t - H_2'F_t + H_2'F_t\|^4 \\
&\le \frac{8}{T}\sum_{t=1}^{T/2}\|\widehat{F}_t - H_1'F_t\|^4 + \frac{8}{T}\sum_{t=1}^{T/2}\|H_1'F_t\|^4 \\
&\quad + \frac{8}{T}\sum_{t=T/2+1}^{T}\|\widehat{F}_t - H_2'F_t\|^4 + \frac{8}{T}\sum_{t=T/2+1}^{T}\|H_2'F_t\|^4 = o_{\mathrm{p}}(1) + \mathcal{O}_{\mathrm{p}}(1), \qquad (\text{F.33})
\end{aligned}$$

where (F.33) is due to (F.8), Lemma B.2 and R.1. □

G Analysis under alternative hypothesis

This appendix provides proofs of the results under the alternative hypothesis as stated in Section 4.

Recall from (4.1) the transformed series Y and write it more compactly by concatenating over time as

$$\begin{pmatrix} Y^b \\ Y^a \end{pmatrix} = \begin{pmatrix} F^b\Lambda_2'\widehat{P}_I^{1\prime} \\ F^a\Lambda_2'\widehat{P}_I^{2\prime} \end{pmatrix} + \begin{pmatrix} \widetilde{\varepsilon}^b \\ \widetilde{\varepsilon}^a \end{pmatrix} := \begin{pmatrix} F^b\widetilde{\Theta}_1' \\ F^a\widetilde{\Theta}_2' \end{pmatrix} + \begin{pmatrix} \widetilde{\varepsilon}^b \\ \widetilde{\varepsilon}^a \end{pmatrix}.$$

Recall also the population model from (4.2) and write it similarly as

$$\begin{pmatrix} Z^b \\ Z^a \end{pmatrix} = \begin{pmatrix} F^b\Lambda_2'P_I^1 \\ F^a\Lambda_2' \end{pmatrix} + \begin{pmatrix} \eta^b \\ \eta^a \end{pmatrix} = \begin{pmatrix} F^b\Theta_1' \\ F^a\Theta_2' \end{pmatrix} + \begin{pmatrix} \eta^b \\ \eta^a \end{pmatrix} = \begin{pmatrix} G^bB' \\ G^aC' \end{pmatrix}\Theta' + \begin{pmatrix} \eta^b \\ \eta^a \end{pmatrix} =: G\Theta' + \eta. \qquad (\text{G.1})$$

To specify Θ in the population model further, we need to distinguish between the two types of breaks as introduced in (4.6) and (4.7) and briefly recalled next.

TA 1: Suppose $\mathrm{col}(\Lambda_1) \cap \mathrm{col}(\Lambda_2) = \{0\}$ and $\mathrm{rk}(\Lambda_1'\Lambda_2) = r$. Then, according to Case *1a*, the alternative model admits a change of Type 1 since $\mathrm{col}(P_I^1\Lambda_2) \cap \mathrm{col}(\Lambda_2) = \{0\}$ and we get

$$\Theta = (\Theta_1, \Theta_2) = \frac{1}{\sqrt{2}}\left(P_I^1\Lambda_2, \Lambda_2\right).$$

TA 2: Suppose $\mathrm{col}(\Lambda_1) \cap \mathrm{col}(\Lambda_2) \neq \{0\}$, $\mathrm{col}(\Lambda_1) \neq \mathrm{col}(\Lambda_2)$ and $\mathrm{rk}(\Lambda_1'\Lambda_2) = r$. Then, according to Case *2a*, the alternative model admits a change of Type 3 since $\mathrm{col}(P_I^1\Lambda_2) \cap \mathrm{col}(\Lambda_2) \neq \{0\}$ and $\mathrm{col}(P_I^1\Lambda_2) \neq \mathrm{col}(\Lambda_2)$. Suppose further that $\Lambda_2 = (\Lambda_1\Phi, \Pi_2)$ with $\Phi \in \mathbb{R}^{r\times r_1}$, and $\Pi_2 \in \mathbb{R}^{d\times r_2}$ with $r_2 = r - r_1$ being linearly independent but not orthogonal to Λ_1. Then, $P_I^1\Lambda_2 = (\Lambda_1\Phi, P_I^1\Pi_2)$ and some (but not all) columns of $\Theta_1 = P_I^1\Lambda_2$ and $\Theta_2 = \Lambda_2$ are linearly independent. Then,

$$\Theta = \left(\Lambda_1, \frac{1}{\sqrt{2}}\Pi_2, \frac{1}{\sqrt{2}}P_I^1\Pi_2\right).$$

The PCA estimators for the alternative models can be derived as in (B.1), that is

$$\widehat{G} = \sqrt{T}\widehat{Q}_{\mathrm{r}_p}, \quad \widehat{\Theta}' = \frac{1}{T}\widehat{G}'Y, \tag{G.2}$$

where $\widehat{Q}_{\mathrm{r}_p}$ are the eigenvectors corresponding to the r_p largest eigenvalues of the $T \times T$ matrix YY'. Furthermore, let $\widehat{\Pi}_{\mathrm{r}_p}$ and $\widehat{V}_{\mathrm{r}_p}$ denote respectively the diagonal matrices with the r_p largest eigenvalues of $\frac{1}{T}YY'$ and $\frac{1}{dT}YY'$. We can also define

$$\bar{G} = \frac{1}{d}Y\bar{\Theta}, \quad \bar{\Theta} = \sqrt{d}\bar{Q}_{\mathrm{r}_p} \tag{G.3}$$

with $\bar{Q}_{\mathrm{r}_p}$ being the r_p eigenvectors corresponding to the r_p largest eigenvalues of the $d \times d$ matrix $Y'Y$.

On the population level as around (A.4), we consider the eigendecomposition

$$\Theta'\Theta = R\Pi_\Theta R',$$

where the diagonal matrix Π_Θ consists of the eigenvalues in decreasing order and the orthogonal matrix R consists of the corresponding eigenvectors. Choose the $d \times \mathrm{r}_p$ matrix

$$Q = \Theta R\Pi_\Theta^{-\frac{1}{2}} \tag{G.4}$$

such that $\Theta\Theta' = Q\Pi_\Theta Q'$ and $Q'Q = I_{\mathrm{r}_p}$.

We follow the same line of proofs as under the null hypothesis. We first provide the proof for the behavior of our test statistic under the alternative hypothesis in Appendix G.1. We then prove auxiliary results in Appendix G.2. Finally, Appendix G.3 states and proves consistency results for the PCA estimators under the alternative models (4.6) and (4.7).

G.1 Proof of result in Section 4

Proof of Proposition 4.1: Proof of (i): The result follows by Propositions G.1 and G.2 below and the same arguments as in the proof of Theorem 4 in Han and Inoue (2015). Write

$$\frac{1}{T}\sum_{t=1}^{T/2}\widehat{G}_t\widehat{G}_t' - \frac{1}{T}\sum_{t=T/2+1}^{T}\widehat{G}_t\widehat{G}_t' = \left(\frac{1}{T}\sum_{t=1}^{T/2}J'G_tG_t'J - \frac{1}{T}\sum_{t=T/2+1}^{T}J'G_tG_t'J\right)$$

$$+\frac{1}{T}\sum_{t=1}^{T/2}\Big(\widehat{G}_t\widehat{G}_t' - J'G_tG_t'J\Big) - \frac{1}{T}\sum_{t=T/2+1}^{T}\Big(\widehat{G}_t\widehat{G}_t' - J'G_tG_t'J\Big). \tag{G.5}$$

The second and third summands in (G.5) can be treated similarly, for instance,

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T/2}\Big(\widehat{G}_t\widehat{G}_t' - J'G_tG_t'J\Big)\\
&= \frac{1}{T}\sum_{t=1}^{T/2}\Big((\widehat{G}_t - J'G_t)(\widehat{G}_t' - G_t'J) + (\widehat{G}_t - J'G_t)G_t'J + JG_t'(\widehat{G}_t' - G_t'J)\Big) = \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\delta_{dT}^2}\Big)
\end{aligned}$$

by Propositions G.1 and G.2.

The asymptotic behavior of the first summand in (G.5) depends on the type of break. We get

$$\frac{1}{T}\sum_{t=1}^{T/2}J'G_tG_t'J - \frac{1}{T}\sum_{t=T/2+1}^{T}J'G_tG_t'J \xrightarrow{\mathrm{p}} J_0'(D_1 - D_2)J_0$$

with D_1, D_2 as in (4.9) and (4.10) and J_0 as the limit of J; see Lemma G.1 below.

Proof of (ii): We refer to the proof of Propostition G.1 for a discussion on the assumptions. The bottom line is that all assumptions are satisfied for the alternative model and we can adopt the proof of Proposition 3.3.

Given all assumptions of Proposition 3.3 are satisfied for the alternative model, we can infer

$$\|\Omega(\widehat{G}) - \Omega(GJ_0)\| \xrightarrow{\mathrm{p}} 0. \tag{G.6}$$

Then,

$$\|\Omega^{-1}(\widehat{G}) - \Omega^{-1}(GJ_0)\| \leq \|\Omega^{-1}(\widehat{G})\|\Big\|\Omega(\widehat{G}) - \Omega(GJ_0)\Big\|\|\Omega^{-1}(GJ_0)\| \xrightarrow{\mathrm{p}} 0 \tag{G.7}$$

by (G.6) and the same arguments as in Q.3 (see (E.4)).

For the test statistic as defined in (3.20), we get

$$\begin{aligned}
W(\widehat{G}) &= V(\widehat{G})'\Omega^{-1}(\widehat{G})V(\widehat{G})\\
&= \frac{T}{b_{T/2}}\frac{1}{\sqrt{T}}V(\widehat{G})'b_{T/2}\Omega^{-1}(\widehat{G})\frac{1}{\sqrt{T}}V(\widehat{G})\\
&= \frac{T}{b_{T/2}}(\mathrm{vech}(\mathcal{C}) + o_{\mathrm{p}}(1))'b_{T/2}\big(\Omega^{-1}(GJ_0) + o_{\mathrm{p}}(1)\big)(\mathrm{vech}(\mathcal{C}) + o_{\mathrm{p}}(1)) \to \infty,
\end{aligned} \tag{G.8}$$

where (G.8) is due to (G.7) and Assumption P.2(i). □

G.2 Proofs of auxiliary results under alternative hypothesis

We state below Propositions G.1 and G.2 which are analogous to Propositions B.1 and B.2 under the null hypothesis. For the statements, we need to introduce further notation including the crucial matrix $\widetilde{J}$ which is analogous to $\widetilde{H}$ in (B.9) under the null hypothesis.

Similarly to $\widehat{F}^b, \widehat{F}^a$, set $\widehat{G}^b = \widehat{G}'_{1:T/2}$ and $\widehat{G}^b = \widehat{G}'_{(T/2+1):T}$ with $\widehat{G}$ as in (G.2). Recall, in particular, that $\widehat{G}$ is a $T \times \mathrm{r}_p$ matrix and r_p differs depending on the type of alternative. We introduce the matrices

$$J_1 := \frac{1}{dT}(\Lambda_2' \widehat{P}_I^{1\prime} \widehat{P}_I^1 \Lambda_2 F^{b\prime} \widehat{G}^b + \Lambda_2' \widehat{P}_I^{1\prime} \widehat{P}_I^2 \Lambda_2 F^{a\prime} \widehat{G}^a) \widehat{V}_{\mathrm{r}_p}^{-1},$$
$$J_2 := \frac{1}{dT}(\Lambda_2' \widehat{P}_I^{2\prime} \widehat{P}_I^2 \Lambda_2 F^{b\prime} \widehat{G}^b + \Lambda_2' \widehat{P}_I^{2\prime} \widehat{P}_I^1 \Lambda_2 F^{a\prime} \widehat{G}^a) \widehat{V}_{\mathrm{r}_p}^{-1}$$

such that

$$\begin{aligned}
\widetilde{G}\widetilde{J} &:= \begin{pmatrix} F^b & 0 \\ 0 & F^a \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} F^b J_1 \\ F^a J_2 \end{pmatrix} \\
&= \frac{1}{dT} \widetilde{G} \begin{pmatrix} \Lambda_2' \widehat{P}_I^{1\prime} \widehat{P}_I^1 \Lambda_2 & \Lambda_2' \widehat{P}_I^{1\prime} \widehat{P}_I^2 \Lambda_2 \\ \Lambda_2' \widehat{P}_I^{2\prime} \widehat{P}_I^1 \Lambda_2 & \Lambda_2' \widehat{P}_I^{2\prime} \widehat{P}_I^2 \Lambda_2 \end{pmatrix} \widetilde{G}' \widehat{G} \widehat{V}_{\mathrm{r}_p}^{-1} \\
&=: \frac{1}{dT} \widetilde{G} \Lambda_P \widetilde{G}' \widehat{G} \widehat{V}_{\mathrm{r}_p}^{-1}.
\end{aligned} \tag{G.9}$$

We write further

$$J = (\Theta'\Theta/d)(G'\widehat{G}/T)\widehat{V}_{\mathrm{r}_p}^{-1}$$

with Θ as in (G.1). Again, Θ differs depending on the respective type of alternative. Note also that $\widetilde{G}$ is the same as $\widetilde{F}$ in (B.9), and so is Λ_P.

The following propositions are analogous to Lemma 10(i) and (ii) in Han and Inoue (2015).

Proposition G.1. *Under Assumptions A.1–A.2, CR.1–CR.4 and P.1,*

$$\frac{1}{T}\|\widehat{G} - GJ\|_F^2 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^2}\right), \text{ as } d, T \to \infty.$$

Proof of Proposition G.1: The proof is analogous to the proof of Proposition B.1 but for the model (G.1) under the alternative. In the proof of Proposition B.1, it is sufficient to impose assumptions on the original models (2.6). The main difference between the proofs under the null hypothesis and the alternative is the following. Under the null hypothesis, the true model we estimate is the same as the one estimated through the transformed series. In contrast, under the alternative, we estimate the parameters in (G.1), which are different from those in the original series X^2, the one we chose to transform. For this reason, the only additional assumption needed is Assumption P.1. Assumption P.1 imposes conditions directly on the loadings of the population model (G.1).

We provide here the beginning of the proof and will then refer to the proof of Proposition B.1. Note first that

$$\frac{1}{T}\|\widehat{G} - GJ\|_F^2 \le \frac{2}{T}\|\widehat{G} - \widetilde{G}\widetilde{J}\|_F^2 + \frac{2}{T}\|\widetilde{G}\widetilde{J} - GJ\|_F^2. \tag{G.10}$$

We consider the two summands in (G.10) separately. For the first summand in (G.10), recall the transformed series $Y_{1:T} = \left(\widehat{P}_I^1 X_{1:T/2}^2, \widehat{P}_I^2 X_{(T/2+1):T}^2\right)$ from (3.7) as well as $Y' := Y_{1:T}$ from (3.15). Due to the PCA estimators (G.2), we get $\frac{1}{dT} YY'\widehat{G} = \widehat{G}\widehat{V}_{\mathrm{r}_p}$ such that $\frac{1}{dT} YY'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1} = \widehat{G}$. Then, given (G.9),

$$\widehat{G} - \widetilde{G}\widetilde{J} = \frac{1}{dT} YY'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1} - \frac{1}{dT}\widetilde{G}\Lambda_P \widetilde{G}'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1}$$

$$= \frac{1}{dT}(YY' - \widetilde{G}\Lambda_P\widetilde{G}')(\widehat{G} - G)\widehat{V}_{\mathrm{r}_p}^{-1} + \frac{1}{dT}(YY' - \widetilde{G}\Lambda_P\widetilde{G}')G\widehat{V}_{\mathrm{r}_p}^{-1}. \tag{G.11}$$

As noted above, with $\widehat{F}$ in (B.9),

$$YY' - \widetilde{G}\Lambda_P\widetilde{G}' = YY' - \widetilde{F}\Lambda_P\widetilde{F}'. \tag{G.12}$$

It was argued in the proof of Proposition B.1 that

$$\|\frac{1}{dT}(YY' - \widetilde{F}\Lambda_P\widetilde{F}')\| = \mathcal{O}_{\mathrm{p}}(1). \tag{G.13}$$

After applying the Frobenius norm and triangle inequality, we consider the two summands in (G.11) separately. For the first summand in (G.11), we get by (G.12),

$$\begin{aligned}
&\frac{1}{T}\|\frac{1}{dT}(YY' - \widetilde{G}\Lambda_P\widetilde{G}')(\widehat{G} - G)\widehat{V}_{\mathrm{r}_p}^{-1}\|_F^2 \\
&\leq \|\frac{1}{dT}(YY' - \widetilde{F}\Lambda_P\widetilde{F}')\|^2\|\widehat{V}_{\mathrm{r}_p}^{-1}\|^2\frac{1}{T}\|\widehat{G} - G\|_F^2 && \text{(G.14)}\\
&= \mathcal{O}_{\mathrm{p}}(1)\mathcal{O}_{\mathrm{p}}(1)\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^2}\right) = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^2}\right), && \text{(G.15)}
\end{aligned}$$

where (G.14) follows by (G.12) and M.1 in Appendix D, and (G.15) follows by (G.13), Proposition G.3 below and Corollary G.1. We also omit the proof for the second summand in (G.11) and refer to the proof of Proposition B.1.

It is left to consider the second summand in (G.10). Recall that $J = (\Theta'\Theta/d)(G'\widehat{G}/T)\widehat{V}_{\mathrm{r}_p}^{-1}$. Then, with further explanations given below, we get

$$\begin{aligned}
&\frac{1}{T}\|\widetilde{G}\widetilde{J} - GJ\|_F^2 \\
&= \frac{1}{T}\frac{1}{(dT)^2}\left\|\widetilde{G}\Lambda_P\widetilde{G}'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1} - G\Theta'\Theta G'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1}\right\|_F^2 \\
&= \frac{1}{T}\frac{1}{(dT)^2}\left\|\widetilde{G}\begin{pmatrix}\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2 & \Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^2\Lambda_2\\ \Lambda_2'\widehat{P}_I^{2'}\widehat{P}_I^1\Lambda_2 & \Lambda_2'\widehat{P}_I^{2'}\widehat{P}_I^2\Lambda_2\end{pmatrix}\widetilde{G}'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1} - G\Theta'\Theta G'\widehat{G}\widehat{V}_{\mathrm{r}_p}^{-1}\right\|_F^2 \\
&\leq \frac{1}{(dT)^2}\left\|\begin{pmatrix}F^b\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2F^{a'} & F^b\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^2\Lambda_2F^{b'}\\ F^a\Lambda_2'\widehat{P}_I^{2'}\widehat{P}_I^1\Lambda_2F^{a'} & F^a\Lambda_2'\widehat{P}_I^{2'}\widehat{P}_I^2\Lambda_2F^{b'}\end{pmatrix} - G\Theta'\Theta G'\right\|^2\frac{1}{T}\|\widehat{G}\|_F^2\|\widehat{V}_{\mathrm{r}_p}^{-1}\|^2 && \text{(G.16)}\\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^2}\right). && \text{(G.17)}
\end{aligned}$$

For (G.17), note first that

$$\begin{pmatrix}F^b\Lambda_2'P_I^{1'}P_I^1\Lambda_2F^{a'} & F^b\Lambda_2'P_I^{1'}P_I^2\Lambda_2F^{b'}\\ F^a\Lambda_2'P_I^{2'}P_I^1\Lambda_2F^{a'} & F^a\Lambda_2'P_I^{2'}P_I^2\Lambda_2F^{b'}\end{pmatrix} = \begin{pmatrix}F^b\Lambda_2'P_I^{1'}\\ F^a\Lambda_2'\end{pmatrix}\begin{pmatrix}P_I^1\Lambda_2F^{a'} & \Lambda_2F^{b'}\end{pmatrix} = G\Theta'\Theta G', \tag{G.18}$$

where we used (G.1). Combining (G.16), (G.18) with M.2, it is left to show the asymptotic behavior for three different summands. We focus on one of them, for example,

$$\begin{aligned}
&\frac{1}{dT}\left\|F^b\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2F^{a'} - F^b\Lambda_2'P_I^{1'}P_I^1\Lambda_2F^{a'}\right\| \\
&\leq \frac{1}{d}\left\|\Lambda_2'\widehat{P}_I^{1'}\widehat{P}_I^1\Lambda_2 - \Lambda_2'P_I^{1'}P_I^1\Lambda_2\right\|\frac{1}{\sqrt{T}}\|F^{a'}\|\frac{1}{\sqrt{T}}\|F^{b'}\|
\end{aligned}$$

$$= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right), \tag{G.19}$$

where the asymptotic of $\frac{1}{\sqrt{T}}\|F^{a'}\|$ is due to S.1 in the proof of Proposition B.1. The behavior of the estimated projection matrices in (G.19) follows under similar considerations as in (B.47).

The asymptotics of $\frac{1}{\sqrt{T}}\|\widehat{G}\|_F$ in (G.16) is due to Proposition G.3 and S.1 in the proof of Proposition B.1. □

Proposition G.2. *Under A.1–A.3(i), CR.1–CR.4 and P.1,*

$$\frac{1}{T^2}\|(\widehat{G} - GJ)'G\|_F^2 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}^4}\right), \text{ as } d, T \to \infty.$$

Proof of Proposition G.2: The proof is analogous to the proof of Proposition B.2 but for the models under the alternative hypothesis in Section 4. We refer to the proof of Proposition G.1 above for the necessary adjustments of the proof of Proposition B.2. □

The following is analogous to Lemma B.2. We omit the proof since it follows similar lines as the proof of Lemma B.2.

Lemma G.1. *Under Assumptions A.1–A.2, CR.1–CR.3 and P.1,*

$$J - J_0 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\delta_{dT}}\right) \quad \text{and} \quad \|J_0\| = \mathcal{O}(1), \text{ as } d, T \to \infty,$$

where, with $\Theta\Theta' = Q\Pi_\Theta Q'$ and $\widetilde{\Pi}$ as in Assumption P.1,

$$J_0 = Q\widetilde{\Pi}Q'\Sigma_G\widetilde{\Pi}^{-1}.$$

G.3 Results for consistency of PCA estimators under alternative hypothesis

This section is analogous to Appendix C but under the alternative hypothesis. We focus on presenting analogues of Lemma C.1 and Proposition C.1. The remaining statements in Appendix C (Lemma C.3 and Corollaries C.1–C.3) are also stated for the alternative models but their proofs are omitted since they follow along the same lines as under the null hypothesis.

Lemma G.2. *Set $\widehat{\Sigma}_Y = \frac{1}{T}Y'Y$. Then, under Assumptions A.1–A.2, CR.1–CR.3 and P.1,*

$$\frac{1}{d}\|\widehat{\Sigma}_Y - \Theta\Theta'\| = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right).$$

Proof: We follow again the proof of Lemma 2 in Doz et al. (2011). Consider

$$\frac{1}{d}\|\widehat{\Sigma}_Y - \Theta\Theta'\| \leq \frac{1}{d}\|\widehat{\Sigma}_Y - \Sigma\| + \frac{1}{d}\|\Sigma - \Theta\Theta'\| \tag{G.20}$$

with

$$\Sigma = \begin{cases} \frac{1}{2}\Theta_1\Theta_1' + \frac{1}{2}\Theta_2\Theta_2' + \Sigma_\eta, & \text{if } \mathit{TA\ 1}, \\ \Lambda_1\Phi\Phi'\Lambda_1' + \frac{1}{2}P_I^1\Pi_2\Pi_2'P_I^{1'} + \frac{1}{2}\Pi_2\Pi_2' + \Sigma_\eta, & \text{if } \mathit{TA\ 2}. \end{cases} \tag{G.21}$$

where

$$\Sigma_\eta = P_I^1\Sigma_\varepsilon P_I^{1'} + P_I^2\Sigma_\varepsilon P_I^{2'}.$$

For the rest of the proof we focus on *TA 2* since it is more involved. *TA 1* follows by similar arguments.

We consider the two summands in (G.20) separately. For the second summand, we get

$$\frac{1}{d}\|\Sigma - \Theta\Theta'\| = \frac{1}{d}\|\Sigma_\eta\| \leq \frac{1}{d}\|P_I^1\|^2\|\Sigma_\varepsilon\| + \frac{1}{d}\|P_I^2\|^2\|\Sigma_\varepsilon\| \leq \frac{2}{d}\|\Sigma_\varepsilon\| \leq O\left(\frac{1}{d}\right) \tag{G.22}$$

due to Assumption CR.3 and our observation (B.18). For the first summand in (G.20), we separate the series according to the transformation (3.7) as

$$\begin{aligned}
\|\widehat{\Sigma}_Y - \Sigma\| &= \left\| \frac{1}{2}\widehat{P}_I^1 \frac{1}{T/2}\sum_{t=1}^{T/2} X_t^2 X_t^{2'} \widehat{P}_I^{1'} + \frac{1}{2}\widehat{P}_I^2 \frac{1}{T/2}\sum_{t=T/2+1}^{T} X_t^2 X_t^{2'} \widehat{P}_I^{2'} - \Sigma \right\| \\
&\leq \frac{1}{2}\left\| \widehat{P}_I^1 \frac{1}{T/2}\sum_{t=1}^{T/2} X_t^2 X_t^{2'} \widehat{P}_I^{1'} - \left(\Lambda_1\Phi\Phi'\Lambda_1' + P_I^1\Pi_2\Pi_2' P_I^{1'} + \Sigma_\eta\right) \right\| \\
&\qquad + \frac{1}{2}\left\| \widehat{P}_I^2 \frac{1}{T/2}\sum_{t=T/2+1}^{T} X_t^2 X_t^{2'} \widehat{P}_I^{2'} - \left(\Lambda_1\Phi\Phi'\Lambda_1' + \Pi_2\Pi_2' + \Sigma_\eta\right) \right\|,
\end{aligned} \tag{G.23}$$

where we used (G.21). We consider the two summands in (G.23) separately. For the first summand in (G.23), with explanations given below,

$$\begin{aligned}
&\left\| \widehat{P}_I^1 \frac{1}{T/2}\sum_{t=1}^{T/2} X_t^2 X_t^{2'} \widehat{P}_I^{1'} - \left(\Lambda_1\Phi\Phi'\Lambda_1' + P_I^1\Pi_2\Pi_2' P_I^{1'} + \Sigma_\eta\right) \right\| \\
&\leq \left\| \widehat{P}_I^1\left(\widehat{\Sigma}_2 - \Lambda_2\Lambda_2'\right)\widehat{P}_I^{1'} \right\| + \left\| \widehat{P}_I^1 \Lambda_2\Lambda_2' \widehat{P}_I^{1'} - \left(\Lambda_1\Phi\Phi'\Lambda_1' + P_I^1\Pi_2\Pi_2' P_I^{1'} + \Sigma_\eta\right) \right\| \\
&\leq \left\| \widehat{P}_I^1\left(\widehat{\Sigma}_2 - \Lambda_2\Lambda_2'\right)\widehat{P}_I^{1'} \right\| + \left\| \widehat{P}_I^1 \Lambda_2\Lambda_2' \widehat{P}_I^{1'} - P_I^1\Lambda_2\Lambda_2' P_I^{1'} \right\| \\
&\qquad + \left\| P_I^1\Lambda_2\Lambda_2' P_I^{1'} - (\Lambda_1\Phi\Phi'\Lambda_1' + P_I^1\Pi_2\Pi_2' P_I^{1'}) \right\| + \|\Sigma_\eta\| \\
&\leq \|\widehat{P}_I^1\|^2\|\widehat{\Sigma}_2 - \Lambda_2\Lambda_2'\| + \left\| \widehat{P}_I^1 \Lambda_2\Lambda_2' \widehat{P}_I^{1'} - P_I^1\Lambda_2\Lambda_2' P_I^{1'} \right\| + \|\Sigma_\eta\| && \text{(G.24)} \\
&\leq \|\widehat{\Sigma}_2 - \Lambda_2\Lambda_2'\| + \left\| \widehat{P}_I^1 \Lambda_2\Lambda_2' \widehat{P}_I^{1'} - P_I^1\Lambda_2\Lambda_2' P_I^{1'} \right\| + \|\Sigma_\eta\| && \text{(G.25)} \\
&= d\mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + d\mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right) + \mathcal{O}(1), && \text{(G.26)}
\end{aligned}$$

where (G.24) uses the submultiplicativity of the spectral norm and $P_I^1\Lambda_2\Lambda_2'P_I^{1'} = \Lambda_1\Phi\Phi'\Lambda_1' + P_I^1\Pi_2\Pi_2'P_I^{1'}$, since $P_I^1\Lambda_2 = (\Lambda_1\Phi, P_I^1\Pi_2)$. Furthermore, (G.25) is due to (B.18) and similarly for their estimated counterparts $\|\widehat{P}_k\| = 1$, $k = 1, 2$, such that $\|\widehat{P}_I^k\| = \frac{1}{2}\|I_d + \widehat{P}_k\| \leq 1$. Finally, (G.26) is due to DGR.1 in Appendix A, similar arguments as in Lemma C.2 and $\|\Sigma_\eta\| = \mathcal{O}(1)$ by the same arguments as in (G.22). The second summand in (G.23) can be handled by analogous considerations. □

Corollary G.1. *Set* $\widehat{\Pi}_{\mathrm{r}_p} = d\widehat{V}_{\mathrm{r}_p}$ *and recall that* $\widehat{V}_{\mathrm{r}_p}$ *is a diagonal matrix consisting of the* r_p *largest eigenvalues of* $\frac{1}{dT}YY'$. *Recall also the diagonal matrix* Π_Θ *from Assumption P.1. Under Assumptions A.1–A.2, CR.1–CR.4 and P.1,*

(i) $\frac{1}{d}\|\widehat{\Pi}_{\mathrm{r}_p} - \Pi_\Theta\| = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right)$,

(ii) $d\|\widehat{\Pi}_{\mathrm{r}_p}^{-1} - \Pi_\Theta^{-1}\| = \mathcal{O}_{\mathrm{p}}\big(\frac{1}{d}\big) + \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\sqrt{T}}\Big)$,

(iii) $\Pi_\Theta\widehat{\Pi}_{\mathrm{r}_p}^{-1} - I_{\mathrm{r}_p} = \mathcal{O}_{\mathrm{p}}\big(\frac{1}{d}\big) + \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\sqrt{T}}\Big)$.

Lemma G.3. *Let $\widehat{A} = (\widehat{a}_{ij})_{i,j=1,\dots,\mathrm{r}_p} = \bar{Q}_{\mathrm{r}_p}' Q$ with Q defined in (G.4) and $\bar{Q}_{\mathrm{r}_p}$ defined in (G.2). Under Assumptions A.1–A.2, CR.1–CR.3 and P.1, $\widehat{a}_{ij} = \mathcal{O}_{\mathrm{p}}\big(\frac{1}{d}\big) + \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\sqrt{T}}\Big)$, $i \neq j$, $\widehat{a}_{ii}^2 = 1 + \mathcal{O}_{\mathrm{p}}\big(\frac{1}{d}\big) + \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\sqrt{T}}\Big)$, $i = 1,\dots,\mathrm{r}_p$.*

Corollary G.2. *Under Assumptions A.1–A.2, CR.1–CR.3 and P.1, one can take $\bar{Q}_{\mathrm{r}_p}$, such that*

$$\bar{Q}_{\mathrm{r}_p}' Q = I_{\mathrm{r}_p} + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right), \quad \|\bar{Q}_{\mathrm{r}_p} - Q\|^2 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{\sqrt{T}}\right).$$

Proposition G.3. *Under Assumptions A.1–A.2, CR.1–CR.3 and P.1,*

$$\frac{1}{T}\|\widehat{G} - G\|_F^2 = \frac{1}{T}\sum_{t=1}^{T}\|\widehat{G}_t - G_t\|_F^2 = \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{T}\right), \text{ as } d, T \to \infty.$$

Proof: Using the representations (G.2), (G.3) and the definition (3.7),

$$\widehat{G}_t = \widehat{V}_{\mathrm{r}_p}^{-\frac{1}{2}}\bar{G}_t = \widehat{V}_{\mathrm{r}_p}^{-\frac{1}{2}}\frac{1}{d}\bar{\Lambda}'Y_t = \widehat{V}_{\mathrm{r}_p}^{-\frac{1}{2}}\frac{1}{\sqrt{d}}\bar{Q}_{\mathrm{r}_p}'Y_t = \begin{cases}\widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1 X_t^2, & \text{for } \ t = 1,\dots,T/2,\\ \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^2 X_t^2, & \text{for } \ t = T/2+1,\dots,T.\end{cases}$$

We consider only the case $t = 1,\dots,T/2$. Then,

$$\begin{aligned}
&\widehat{G}_t - G_t\\
&= \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1 X_t^2 - G_t\\
&= \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1(\Lambda_2 F_t + \varepsilon_t) - G_t\\
&= \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'(\widehat{P}_I^1\Lambda_2 - P_I^1\Lambda_2)F_t + \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'P_I^1\Lambda_2 F_t - G_t + \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1\varepsilon_t\\
&= \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'(\widehat{P}_I^1\Lambda_2 - P_I^1\Lambda_2)F_t + (\widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\Theta - I_{\mathrm{r}_p})G_t + \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1\varepsilon_t \qquad \text{(G.27)}\\
&= \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_r'\frac{1}{2}(\widehat{P}_1\Lambda_2 - P_{0,1}\Lambda_2)F_t + \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}(\bar{Q}_{\mathrm{r}_p}'Q - \widehat{\Pi}_{\mathrm{r}_p}^{1/2}\Pi_\Theta^{-1/2})\Pi_\Theta^{1/2}G_t + \widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\widehat{P}_I^1\varepsilon_t, \qquad \text{(G.28)}
\end{aligned}$$

where (G.27) follows by (G.1). We consider the three summands in (G.28) separately. For the first one, note that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T/2}\|\widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\frac{1}{2}(\widehat{P}_1\Lambda_2 - P_{0,1}\Lambda_2)F_t\|_F^2 &\leq \|\widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\bar{Q}_{\mathrm{r}_p}'\frac{1}{2}(\widehat{P}_1\Lambda_2 - P_{0,1}\Lambda_2)\|^2\frac{1}{T}\|F\|_F^2\\
&\leq d\|\widehat{\Pi}_{\mathrm{r}_p}^{-1/2}\|^2\|\bar{Q}_{\mathrm{r}_p}'\|^2\frac{1}{d}\|\frac{1}{2}(\widehat{P}_1\Lambda_2 - P_{0,1}\Lambda_2)\|^2\frac{1}{T}\|F\|_F^2\\
&= \mathcal{O}_{\mathrm{p}}\left(\frac{1}{d^2}\right) + \mathcal{O}_{\mathrm{p}}\left(\frac{1}{T}\right),
\end{aligned}$$

since $\frac{1}{T}\|F\|_F^2 \leq \frac{r}{T}\|F\|^2 = \mathcal{O}_{\mathrm{p}}(1)$ by S.1 in the proof of Proposition B.1, $\widehat{\Pi}_{\mathrm{r}_p}^{-1/2} = \frac{1}{\sqrt{d}}\Big(\frac{1}{d}\widehat{\Pi}_{\mathrm{r}_p}\Big)^{-1/2} = \mathcal{O}_{\mathrm{p}}\Big(\frac{1}{\sqrt{d}}\Big)$ by Corollary G.1 and the asymptotic of $\frac{1}{d}\|\frac{1}{2}(\widehat{P}_1\Lambda_2 - P_{0,1}\Lambda_2)\|^2$ can be proved as in (C.15).

We omit the remainder of the proof concerning the other two terms in (G.28) since it follows the exact same arguments as the proof of Proposition C.1. □

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