Scaling limits for sample autocovariance operators of Hilbert space-valued linear processes*

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Abstract

This article considers linear processes with values in a separable Hilbert space exhibiting long-range dependence. The scaling limits for the sample autocovariance operators at different time lags are investigated in the topology of their respective Hilbert spaces. Distinguishing two different regimes of long-range dependence, the limiting object is either a Hilbert space-valued Gaussian or a Hilbert space-valued non-Gaussian random variable. The latter can be represented as a unitary transformation of double Wiener-Itô integrals with sample paths in a function space. This work is the first to show weak convergence to such double stochastic integrals with sample paths in infinite dimensions. The result generalizes the well known convergence to a Hermite process in finite dimensions, introducing a new domain of attraction for probability measures in Hilbert spaces. The key technical contributions include the introduction of double Wiener-Itô integrals with values in a function space and with dependent integrators, as well as establishing sufficient conditions for their existence as limits of sample autocovariance operators.

Keywords: Linear processes, double stochastic integrals, autocovariance operators, long-range dependence, Rosenblatt distribution, functional data analysis.

1 Introduction

In this article, we investigate the weak convergence of the sample autocovariance operators of a Hilbert space-valued linear process exhibiting long-range dependence. Our setting is as follows: Let $\mathbb{H}$ denote a separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\| \cdot \|_{\mathbb{H}}$. We further write $L(\mathbb{H})$ for the set of all bounded linear operators on $\mathbb{H}$. We consider a sequence of random variables $\{X_n\}_{n\in\mathbb{Z}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with values in $\mathbb{H}$. Suppose the stochastic process $\{X_n\}_{n\in\mathbb{Z}}$ admits the linear representation

$$X_n = \sum_{j=0}^{\infty} u_j[\varepsilon_{n-j}], \quad n \in \mathbb{Z}, \tag{1.1}$$

with $u_j \in L(\mathbb{H})$ for all $j \in \mathbb{N}$ and $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is a sequence of $\mathbb{H}$-valued independent, identically distributed (i.i.d.), zero-mean random variables.

For a given time lag $h \in \mathbb{N}_0$, the sample autocovariance operator $\widehat{\Gamma}_{N,h}$ and its population counterpart Γ_h of a process $\{X_n\}_{n\in\mathbb{Z}}$ are given by

$$\widehat{\Gamma}_{N,h} \doteq \frac{1}{N} \sum_{n=1}^{N} X_{n+h} \otimes X_n \quad \text{and} \quad \Gamma_h \doteq \mathrm{E}(X_h \otimes X_0). \tag{1.2}$$

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Both quantities in (1.2) take values in $\mathbb{H}^{\otimes 2}$. We are interested in the weak convergence of the normalized operators

$$A_N((\widehat{\Gamma}_{N,h} - \Gamma_h), h = 0, \ldots, H), \quad \text{as } N \to \infty, \tag{1.3}$$

in $(\mathbb{H}^{\otimes 2})^{\times(H+1)}$, where $\{A_N\}_{N \in \mathbb{N}} \in L((\mathbb{H}^{\otimes 2})^{\times(H+1)})$ is a sequence of suitable scaling operators.

The asymptotic behavior of (1.2)–(1.3) is by now well understood when $\{X_n\}$ in (1.1) (for general $\{u_j\}_{j\in\mathbb{N}_0}$) exhibits Short-Range Dependence (SRD), i.e., when $\sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}} < \infty$, where $\|\cdot\|_{\mathrm{op}}$ denotes the usual operator norm recalled in (2.3) below. In this case, the normalizing sequence is simply $A_N = N^{1/2}$ and the quantity in (1.3) satisfies a Central Limit Theorem (CLT) in $\mathbb{H} \otimes \mathbb{H}$ (or, equivalently, the space of Hilbert-Schmidt operators on $\mathbb{H}$), i.e., converges to a centered Gaussian random variable taking values in $\mathbb{H} \otimes \mathbb{H}$, with an explicit covariance operator; see Mas (2002) and (3.4)–(3.6) below.

In contrast to Mas (2002), we are interested in the case where $\{X_n\}$ in (1.1) exhibits Long-Range Dependence (LRD), in the sense that,

$$\sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}} = \infty. \tag{1.4}$$

Under (1.4), the fluctuations of the sample autocovariance operator (1.3) crucially depend on the convergence or divergence of the series $\sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}}^{4/3}$. When this series converges, we obtain a CLT for the sample autocovariance operators for general $\{u_j\}_{j\in\mathbb{N}_0}$. When it diverges, in addition to (1.1), we need to impose the additional structure on $\{u_j\}_{j\in\mathbb{N}_0}$ given by

$$u_j \doteq (j+1)^{T-I}, \ j \in \mathbb{N}, \tag{1.5}$$

where I is the identity operator in $\mathbb{H}$ and $T \in L(\mathbb{H})$ is a self-adjoint operator; see Section 2.3 below for more details. In this case, both the usual scaling $A_N = N^{1/2}$ and the Gaussian limit are lost.

Unless otherwise stated and to simplify the presentation, we assume for the rest of the introduction that $\mathbb{H}$ is the space of square integrable functions with regard to a σ-finite measure μ, i.e., $\mathbb{H} = L^2(\mathbb{Y}, \mathcal{A}, \mu)$. We assume, moreover, that $T = D_d$ is a multiplication operator associated to a measurable function d, evaluated by $D_d f \doteq \{d(y)f(y), y \in \mathbb{Y}\}$ for all $f \in L^2(\mathbb{Y}, \mathcal{A}, \mu)$. Then, (1.5) is recast as

$$u_j \doteq (j+1)^{D_d - I}. \tag{1.6}$$

When $d(s) \in \left(0, \frac{1}{2}\right)$ for $s \in \mathbb{Y}$, $\{u_j\}$ in (1.6) satisfies (1.4), which justifies referring to (1.1) with (1.6) and $d(s) \in \left(0, \frac{1}{2}\right)$ as the LRD case. Therefore, the results of Mas (2002), that require absolute summability of $\{u_j\}$ in the operator norm, are no longer applicable.

The sample mean for LRD case (1.1) with (1.6) (i.e., $\mathbb{H} = L^2(\mathbb{Y}, \mathcal{A}, \mu)$) was studied in Račkauskas and Suquet (2010); Račkauskas and Suquet (2011). The authors derived a Gaussian limiting distribution for the sample mean and the piecewise linear functions associated with the partial sums. Their results were generalized by Düker (2018) to LRD processes with values in a general Hilbert space $\mathbb{H}$ and representation (1.1) with (1.5). The fluctuations of the sample autocovariance operator for this model have not been studied, and are the main objective of the present paper.

LRD has been studied extensively for stochastic processes in finite dimensions; see Giraitis, Koul, and Surgailis (2012); Beran, Feng, Ghosh, and Kulik (2013); Pipiras and Taqqu (2017) to name a few. Special to long-range dependence is the Rosenblatt process, a non-Gaussian process that can be represented as a double Wiener-Itô integral and arises as a limiting object for stationary processes exhibiting long-range dependence; see Taqqu (1975); Rosenblatt (1979) for some of the earlier references and Tudor (2008); Veillette and Taqqu (2013); Leonenko, Ruiz-Medina, and Taqqu (2017); Bai and Taqqu (2017) to name a few more recent ones.

The weak convergence of the (suitably re-normalized) sample autocovariance estimators in (1.2)–(1.3) when the underlying model (1.1) with (1.6) takes values in $\mathbb{H} = \mathbb{R}$ (and hence has a constant memory parameter d) was studied in Horváth and Kokoszka (2008). There, it was shown that the scaling limit is a Brownian motion with $A_N = N^{1/2}$ when $d \in \left(0, \frac{1}{4}\right)$, and follows the Rosenblatt distribution with $A_N = N^{1-2d}$ when $d \in \left(\frac{1}{4}, \frac{1}{2}\right)$. In Düker (2020), these results were extended to multivariate linear processes, allowing some components to exhibit short- and others long-range dependence.

In analogy to Horváth and Kokoszka (2008), the two regimes $d \in \left(0, \frac{1}{4}\right)$ and $d \in \left(\frac{1}{4}, \frac{1}{2}\right)$ for the fluctuations of the autocovariance operator roughly correspond to the two regimes $d(s) \in \left(0, \frac{1}{4}\right)$ and $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right)$ for each $s \in \mathbb{Y}$ in our case. We prove weak convergence of (1.2) to a Gaussian law in $L^2(\mathbb{Y}, \mathcal{A}, \mu)$ for the first regime, and to a double Wiener-Itô integral with sample paths in $L^2(\mathbb{Y}, \mathcal{A}, \mu)$ for the second regime. For the first regime, we extend the arguments of Mas (2002). The second regime requires a careful analysis of double Wiener-Itô integrals with sample paths in Hilbert spaces and with spatially dependent integrators. This substantially extends results from Fox and Taqqu (1985) and Norvaiša (1994); see Section 4. Establishing the tools that prove convergence to a non-Gaussian limit in functional spaces is one of the key challenges to be addressed in this work; see Lemma 6.3. To the best of our knowledge, this is the first instance of convergence to a double Wiener-Itô integral with sample paths in a Hilbert space.

Our work is closely related to functional data analysis, where time series in infinite dimensions exhibiting LRD have appeared in numerous domains such as finance; see, e.g., Cajueiro and Tabak (2005); Alvarez-Ramirez, Alvarez, Rodriguez, and Fernandez-Anaya (2008); Preciado and Morris (2008); Casas and Gao (2008). It is therefore reasonable to model such problems using $\{X_n\}_{n\in\mathbb{Z}}$ as in (1.1) with (1.5). Limit theorems such as those developed in this work are important for applications, as they can be used to design hypothesis tests and construct confidence bands. We emphasize that, although there are many works dealing with modeling and theoretical aspects of functional time series exhibiting SRD (e.g., Merlevède, Peligrad, and Utev (1997); Jirak (2018); Mas (2002); Düker and Zoubouloglou (2024); Rademacher, Kreiß, and Paparoditis (2024)), such works are comparatively scarce in the context of LRD. Notable exceptions include the works of Characiejus and Račkauskas (2013, 2014); Düker (2018); Li, Robinson, and Shang (2020); Ruiz-Medina (2022); Durand and Roueff (2024). For further details on functional data analysis, we refer to Hsing and Eubank (2015); Hörmann, Kokoszka, and Nisol (2018), while Bosq (2000) remains the canonical reference for linear processes with values in Banach spaces.

The rest of the paper is organized as follows. In Section 2, we introduce some notation and recall some preliminary technical results. In Section 3, we present our main results on the scaling limits of the sample autocovariance operators of the process $\{X_n\}_{n\in\mathbb{Z}}$. In Section 4, we introduce double Wiener-Itô integrals with values in a function space and spatially dependent integrators; such objects are crucial in representing the resulting limit in the second regime. In Section 5, we present the proofs of our main results. Section 6 is concerned with some technical lemmas and their proofs.

2 Preliminaries

In this section, we introduce notation, collect some preliminary facts on operators and function spaces used throughout the paper, and give some properties of the linear process (1.1).

2.1 Notation and terminology

Let $(X, \mathcal{G}, \nu)$ be a σ-finite measure space and $(Y, \|\cdot\|_Y)$ a normed space. Then $L^2(X : Y) \doteq L^2(X, \mathcal{G}, \nu : Y)$ denotes the space of square integrable, measurable functions on $(X, \mathcal{G}, \nu)$, taking values in Y. When $(Y, \|\cdot\|_Y) = (\mathbb{R}, |\cdot|_{\mathbb{R}})$, we write $L^2(X)$ or $L^2(X, \nu)$. Recall that $L^2(X)$ is equipped with the inner product

$$\langle f, g\rangle_{L^2(X)} = \int_X f(s)g(s)\nu(ds), \quad f, g \in L^2(X),$$

and its induced norm $\|\cdot\|_{L^2(X)}$. Moreover, denote, for a fixed integer $m \geqslant 1$, the space $L^2(X^{\times m}) \doteq L^2(X^{\times m}, \mathcal{G}^{\otimes m}, \nu^{\otimes m})$ of square integrable real-valued functions in the product measure space $(X^{\times m}, \mathcal{G}^{\otimes m}, \nu^{\otimes m})$.

For $a, b > 0$, define the beta function

$$\mathrm{B}(a, b) \doteq \int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^\infty x^{a-1}(x+1)^{-(a+b)}dx, \quad a, b > 0. \tag{2.1}$$

Moreover, for a given function $d : \mathbb{Y} \to \mathbb{R}_+$, we define two functions $c : \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}$ and $c : \mathbb{Y} \to \mathbb{R}$ by

$$c(r, s) \doteq \mathrm{B}(d(r), d(s)) = \int_0^\infty x^{d(r)-1}(x+1)^{d(s)-1}dx, \quad \text{and} \quad c(r) \doteq c(r, r), \tag{2.2}$$

respectively.

Throughout this work, we write $(\mathbb{R}, \mathcal{B}, \lambda)$ to denote the real numbers, equipped with the Lebesgue measure λ, and the Borel sets (in the usual topology) $\mathcal{B}$.

2.2 Elements of operator theory

Let $L(X, Y)$ be the space of bounded linear operators from X to Y, and let $L(X) \doteq L(X, X)$. The space $(L(\mathbb{H}), \|\cdot\|_{\mathrm{op}})$ forms a Banach space, with the operator norm defined by

$$\|T\|_{\mathrm{op}} \doteq \inf\{c \geqslant 0 : \|Tv\|_{\mathbb{H}} \leqslant c\|v\|_{\mathbb{H}} \text{ for all } v \in \mathbb{H}\}, \ T \in L(\mathbb{H}). \tag{2.3}$$

The sample autocovariance operators are considered to be random elements with values in the space of Hilbert-Schmidt operators on $\mathbb{H}$, denoted by $\mathrm{HS}(\mathbb{H})$. A Hilbert-Schmidt operator $A : \mathbb{H} \to \mathbb{H}$ is a bounded operator with finite Hilbert-Schmidt norm

$$\|A\|^2_{\mathrm{HS}(\mathbb{H})} \doteq \sum_{i=1}^\infty \|Ae_i\|^2_{\mathbb{H}} < \infty, \tag{2.4}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$. The space $\mathrm{HS}(\mathbb{H})$ equipped with the inner product $\langle A, B\rangle_{\mathrm{HS}(\mathbb{H})} = \sum_{i=1}^\infty \langle Ae_i, Be_i\rangle_{\mathbb{H}}$ and its induced norm $\|A\|_{\mathrm{HS}(\mathbb{H})}$ is a separable Hilbert space itself. Recall the (isometric) isomorphism $\mathbb{H} \otimes \mathbb{H} \cong \mathrm{HS}(\mathbb{H})$; see, e.g., Muandet, Fukumizu, Sriperumbudur, and Schölkopf (2017), pp. 32-33. Invoking this isomorphism, we either show convergence of the sample autocovariance operators in $\mathrm{HS}(\mathbb{H})$ (Theorem 3.1 and Corollary 3.2) or $\mathbb{H} \otimes \mathbb{H}$ (Theorems 3.3 and 3.5), but think of weak convergence as equivalent in $\mathbb{H} \otimes \mathbb{H}$ and $\mathrm{HS}(\mathbb{H})$.

Closely related is the Banach space of trace class operators, denoted by $\mathrm{Tr}(\mathbb{H})$, and equipped with the norm

$$\|T\|_{\mathrm{Tr}(\mathbb{H})} = \sum_{i=1}^\infty \langle |T|u_i, u_i\rangle_{\mathbb{H}}, \tag{2.5}$$

with $|T| = \sqrt{T^*T}$, where T^* denotes the adjoint of the operator T. Whenever the space $\mathbb{H}$ is easily inferred from context, we may write Tr instead of Tr($\mathbb{H}$). If T is a non-negative, self-adjoint operator (i.e., $T = T^*$), then $\|T\|_{\mathrm{Tr}(\mathbb{H})} = \mathrm{Tr}(T) = \sum_{i=1}^{\infty}\langle Tu_i, u_i\rangle_{\mathrm{Tr}(\mathbb{H})}$. Examples of non-negative, self-adjoint operators are covariance operators. The three norms (2.3)–(2.5) satisfy

$$\|\cdot\|_{\mathrm{op}} \leqslant \|\cdot\|_{\mathrm{HS}(\mathbb{H})} \leqslant \|\cdot\|_{\mathrm{Tr}(\mathbb{H})}.$$

Let $(\mathbb{Y}, \mathcal{A}, \mu)$ be a separable, σ-finite measure space. Then, the following isomorphisms hold,

$$L^2(\mathbb{Y} \times \mathbb{Y}, \mu \otimes \mu) \cong L^2(\mathbb{Y}, \mu) \otimes L^2(\mathbb{Y}, \mu) \cong L^2(\mathbb{Y}, \mu : L^2(\mathbb{Y}, \mu)).$$

Let $T \in L(\mathbb{H})$ be a self-adjoint operator, and recall that a unitary operator U is such that $UU^* = U^*U = I$, where I is the identity operator. The spectral theorem for self-adjoint operators states that each self-adjoint operator is decomposable into a unitary operator and a multiplication operator; see Theorem 9.4.6 in Comway (1994). More precisely, there exist a measure space $(\mathbb{Y}, \mathcal{A}, \mu)$, a unitary operator U and a multiplication operator D_d on $(\mathbb{Y}, \mathcal{A}, \mu)$ associated to a bounded function d, such that

$$UTU^* = D_d, \quad U : \mathbb{H} \to L^2(\mathbb{Y}), \quad D_d : L^2(\mathbb{Y}) \to L^2(\mathbb{Y}). \tag{2.6}$$

Moreover, since $\mathbb{H}$ is separable, the measure space $(\mathbb{Y}, \mathcal{A}, \mu)$ is σ-finite; see Proposition 9.4.7 in Comway (1994). The multiplication operator D_d is given by

$$D_d[f](s) \doteq d(s)f(s), \quad f \in L^2(\mathbb{Y}),\ s \in \mathbb{Y}. \tag{2.7}$$

2.3 Properties of the linear process

We collect here some general properties of linear processes with representation (1.1) as well as implications of imposing (1.5), where $T \in L(\mathbb{H})$ is a self-adjoint operator. With regard to stochastic processes with sample paths in function spaces, we will make no distinction between the class of square integrable functions $\mathcal{L}^2$ and its respective family of equivalence classes L^2; see the first paragraph of Section 3 in Characiejus and Račkauskas (2013).

The linear process $\{X_n\}_{n\in\mathbb{N}}$ in (1.1) converges P-almost surely and in $L^2(\Omega : \mathbb{H})$ if $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^2 < \infty$, $\mathrm{E}\,\varepsilon_0 = 0$ and $\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^2 < \infty$; see Lemma 7.1 in Bosq (2000). This implies that series that are SRD in the sense that $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}} < \infty$ converge. Moreover, it shows that LRD series in the sense that $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^{4/3} < \infty$ convergence. However, for series that are LRD (in the sense of (1.4)), both cases $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^2 < \infty$ and $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^2 = \infty$ are possible. For an instance of the latter case, note that with $\{u_j\}$ satisfying (1.6), we have $\operatorname{ess\,sup}_{s\in\mathbb{Y}} d(s) = 1/2$ if and only if $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^2 = \infty$. In this case, the well posedness of the process is not given by the results in Bosq (2000) and requires some additional assumptions. We first address the well-posedness of the linear processes $\{X_n\}_{n\in\mathbb{Z}}$ in (1.1) when $\sum_{j=0}^{\infty}\|u_j\|_{\mathrm{op}}^2 = \infty$ for the case $\mathbb{H} = L^2(\mathbb{Y})$. We then leverage the well-posedness on $L^2(\mathbb{Y})$ and the spectral theorem (2.6), to derive the convergence of an $\mathbb{H}$-valued linear process (1.1)–(1.5) for a general $\mathbb{H}$.

First consider $\mathbb{H} = L^2(\mathbb{Y})$ and rewrite the model (1.1) with (1.6) as

$$X_n(r) = \sum_{j=0}^{\infty}(j+1)^{d(r)-1}\varepsilon_{n-j}(r), \quad r \in \mathbb{Y}, \tag{2.8}$$

where $d(r) \in \left(0, \frac{1}{2}\right)$ for all $r \in \mathbb{Y}$ and $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is an $L^2(\mathbb{Y})$-valued i.i.d. sequence with

$$\sigma(r,s) \doteq \mathrm{E}(\varepsilon_0(r)\varepsilon_0(s)), \quad \sigma^2(r) \doteq \mathrm{E}\,|\varepsilon_0(r)|^2, \quad r, s \in \mathbb{Y}. \tag{2.9}$$

The stochastic process $\{X_n\}$ in (2.8) (equivalently, (1.1) with (1.6)) has sample paths that belong to $L^2(\mathbb{Y})$ a.s. if the following condition is satisfied

$$\int_{\mathbb{Y}} \frac{\sigma^2(r)}{1-2d(r)}\mu(dr) < \infty, \quad \text{implying that } \mathrm{E}\,\|\varepsilon_0\|^2_{L^2(\mathbb{Y})} = \int_{\mathbb{Y}} \sigma^2(r)\mu(dr) < \infty; \tag{2.10}$$

see Proposition 3 in Characiejus and Račkauskas (2013). Note that Characiejus and Račkauskas (2013) use a different notation $u_j = (j+1)^{-\tilde{d}(r)}$ with $\tilde{d}(r) = 1 - d(r)$. In (2.10), their conditions are adjusted to our setting. Furthermore, the $L^2(\mathbb{Y})$-valued series (2.8) converges in mean square and P-almost surely if $d(s) < \frac{1}{2}$ μ-a.s. under the second condition in (2.10); see Proposition 4 in Characiejus and Račkauskas (2014), again after adjusting to the different notation. Hence Condition (2.10) ensures the well-posedness of (2.8) when $\operatorname{ess\,sup}_{s\in\mathbb{Y}} d(s) = 1/2$.

We now treat the case of a general $\mathbb{H}$. Invoking the spectral theorem, one can infer that (1.1) with (1.5) also converges P-a.s. if $d(s) < \frac{1}{2}$ μ-a.s., where d is the function associated to the multiplication operator D_d in the decomposition of the spectral theorem (2.6)–(2.7). We refer to p. 1445 in Düker (2018) and calculations done in (5.36)–(5.37) below.

Moreover, after accounting for the aforementioned change in notation, a CLT for the sample mean of $\{X_n\}_{n\in\mathbb{Z}}$ with values in $L^2(\mathbb{Y})$ was shown in Proposition 4 of Characiejus and Račkauskas (2013) under the conditions

$$\int_{\mathbb{Y}} \frac{\sigma^2(r)}{d^2(r)}\mu(dr) < \infty, \quad \int_{\mathbb{Y}} \frac{\sigma^2(r)}{d(r)(1-2d(r))}\mu(dr) < \infty. \tag{2.11}$$

In the next section we compare these conditions with the corresponding ones for the weak convergence of the autocovariance operators.

For the sample autocovariances of $\{X_n\}_{n\in\mathbb{Z}}$ in (2.8) and its corresponding population quantities, we recast (1.2) as

$$\widehat{\gamma}_{N,h}(r,s) \doteq \frac{1}{N}\sum_{n=1}^{N} X_{n+h}(r)X_n(s) \quad \text{and} \quad \gamma_h(r,s) \doteq \mathrm{E}(X_h(r)X_0(s)). \tag{2.12}$$

Note that henceforth, we use the notations $\widehat{\Gamma}_{N,h}, \Gamma_h$ for the autocovariance operators in a general space $\mathbb{H}$, and their lowercase counterparts $\widehat{\gamma}_{N,h}, \gamma_h$ for the special case $\mathbb{H} = L^2(\mathbb{Y})$.

Finally, recall that, since our linear series admit second moments, the covariance operators are nuclear operators and therefore Hilbert-Schmidt; see p. 6 in Bosq (2000). This remains true for the empirical covariance operator that belongs P-almost surely to the space $\mathbb{H}\otimes\mathbb{H} \cong \mathrm{HS}(\mathbb{H})$; see pp. 36–37 in Bosq (2000).

Remark 2.1. 1. *To ensure that* (1.5) *indeed implies long-range dependence in the sense of* (1.4)*, we employ the spectral decomposition* (2.6)*. Then,* (1.4) *holds if and only if* $\operatorname{ess\,sup}_{s\in Y} d(s) \geqslant 0$*; see p. 1445 in Düker (2018).*

2. *Recall that, for $\mathbb{H} = \mathbb{R}$, LRD is often modeled through a linear process with $u_j = j^{d-1}\ell(j)$ for $d \in (0, \frac{1}{2})$, where $\ell(j)$ is a slowly varying function; see Condition I on p. 17 in Pipiras and Taqqu (2017). The slowly varying function ℓ induces flexibility on the sequence u_j. While* (1.6) *naturally generalizes the real-valued model to the Hilbert space-valued setting allowing for a space-varying memory parameter, it is quite restrictive as a function in j. We emphasize that our results (Theorems 3.1 and 3.5) can be generalized to using $u_j = (j+1)^{D_d - I}\ell(j)$ with ℓ being a slowly varying function instead. Since the function ℓ is real-valued, one can adjust our proofs by incorporating the arguments for real-valued linear processes; see Chapter 2 in Pipiras and Taqqu (2017). For the sake of clarity, we only consider here the case $\ell(j) = 1$ and focus on the remaining technical difficulties.*

3 Main Results

In this section, we introduce the limiting objects and state the convergence results for the sample autocovariance operators (1.2). From here on we distinguish the following two cases, roughly corresponding to the two regimes $d(s) \in \left(0, \frac{1}{4}\right)$ and $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right)$ for each $s \in \mathbb{Y}$. First, suppose that $\{u_j\}_{j \in \mathbb{N}_0}$ is such that

$$\sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}}^{4/3} < \infty, \tag{3.1}$$

which in the following we refer to as *first regime*. Second, suppose that T is a self-adjoint operator and that

$$u_j = (j+1)^{T-I}, \quad T = U D_d U^* \quad \text{with} \quad d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right), \; s \in \mathbb{Y}, \tag{3.2}$$

which we refer to from now on as *second regime*. In (3.2), U, D_d are as in the spectral theorem (2.6). In particular, under the second regime, the series in (3.1) diverges.

We start by defining the covariance operator of the limiting Gaussian process in the first regime. We view the limiting Gaussian element for a single time lag as an element of $\mathrm{HS}(\mathbb{H})$. For the joint convergence of autocovariance operators across time lags $0, 1, \ldots, H$, for $H \in \mathbb{N}$, the corresponding Gaussian limit is an element of $\mathrm{HS}(\mathbb{H})^{\times(H+1)}$, and its covariance operator is an element of trace class on the space $\mathrm{HS}(\mathbb{H})^{\times(H+1)}$. It can be identified with an operator taking the block form

$$\Sigma \doteq (\Sigma_{\Gamma}^{(p,q)})_{p,q=0,\ldots,H}. \tag{3.3}$$

For $p, q = 0, \ldots, H$, and $T \in \mathrm{HS}(\mathbb{H})$, the cross-covariance operator $\Sigma_{\Gamma}^{(p,q)}$ is given by

$$\Sigma_{\Gamma}^{(p,q)}(T) = \sum_{m=0}^{\infty} \Gamma_{m+p-q} T \Gamma_m + \sum_{m=0}^{\infty} \Gamma_{m+q} T \Gamma_{m-p} + A_q(\Lambda - \Phi) A_p(T), \tag{3.4}$$

where, upon recalling that $\langle \varepsilon_0, \cdot \rangle_{\mathbb{H}} \varepsilon_0 \in \mathrm{HS}(\mathbb{H})$, we have $\Lambda, \Phi \in \mathrm{HS}\left(\mathrm{HS}(\mathbb{H})\right)$ with

$$\Lambda(T) \doteq \mathrm{E}\left(\langle \langle \varepsilon_0, \cdot \rangle_{\mathbb{H}} \varepsilon_0, T \rangle_{\mathrm{HS}(\mathbb{H})} (\langle \varepsilon_0, \cdot \rangle_{\mathbb{H}} \varepsilon_0)\right), \quad \Phi(T) \doteq \langle C, T + T^* \rangle_{\mathrm{HS}(\mathbb{H})} C + \langle C, T \rangle_{\mathrm{HS}(\mathbb{H})} C, \tag{3.5}$$

and $C, \Gamma_h \in \mathrm{HS}(\mathbb{H})$, $A_p \in \mathrm{HS}\left(\mathrm{HS}(\mathbb{H})\right)$ with

$$C \doteq \mathrm{E}\left(\langle \varepsilon_0, \cdot \rangle_{\mathbb{H}} \varepsilon_0\right), \quad \Gamma_h \doteq \mathrm{E} \langle X_h, \cdot \rangle_{\mathbb{H}} X_0, \quad A_p(T) \doteq \sum_{j \in \mathbb{N}} u_{j+p} T u_j^*. \tag{3.6}$$

Note that the two representations of Γ_h in (1.2) and (3.6) are due to the aforementioned isomorphism $\mathbb{H} \otimes \mathbb{H} \cong \mathrm{HS}(\mathbb{H})$. The representation (3.4)–(3.6) is due to Lemma 3 in Mas (2002), but the notation is slightly changed to be consistent with the present paper.

Theorem 3.1 (First regime)**.** *Let $\{X_n\}_{n \in \mathbb{Z}}$ be an $\mathbb{H}$-valued linear process* (1.1)*, and consider its autocovariance operators* ($\widehat{\Gamma}_{N,h}, h = 0, \ldots, H$) *given in* (1.2) *with $\widehat{\Gamma}_{N,h} \in \mathrm{HS}(\mathbb{H})$ for all h with a slight abuse of notation. Suppose* (3.1) *and*

$$\mathrm{E}\, \|\varepsilon_0\|_{\mathbb{H}}^4 < \infty. \tag{3.7}$$

Then,

$$\sqrt{N} \begin{pmatrix} \widehat{\Gamma}_{N,0} - \Gamma_0 \\ \vdots \\ \widehat{\Gamma}_{N,H} - \Gamma_H \end{pmatrix} \xrightarrow{d} G \doteq \begin{pmatrix} G_0 \\ \vdots \\ G_H \end{pmatrix} \in (\mathrm{HS}(\mathbb{H}))^{\times(H+1)}, \tag{3.8}$$

where the weak convergence holds in the topology of $(\mathrm{HS}(\mathbb{H}))^{\times(H+1)}$, and G is the centered Gaussian element of $\mathrm{HS}(\mathbb{H})^{\otimes(H+1)}$ with covariance operator Σ given in (3.3)*–*(3.6)*.*

The proof of Theorem 3.1 is postponed to Section 5.1. Note that Theorem 3.1 does not require any structural assumptions on the sequence $\{u_j\}_{j\in\mathbb{N}_0}$ besides (3.1). In particular, (1.5) is not needed. The following corollary states the implications of Theorem 3.1 on the linear series with values in $L^2(\mathbb{Y})$ with (1.5), as written in (2.8).

Corollary 3.2. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be an $L^2(\mathbb{Y})$-valued process (2.8) and consider its autocovariance operators $(\widehat{\gamma}_{N,h}, h = 0, \dots, H)$ given in (2.12), where we consider $\widehat{\gamma}_{N,h}$ as an element of $\mathrm{HS}(L^2(\mathbb{Y}))$ for all h, with a slight abuse of notation. Suppose $\operatorname{ess\,sup}_{s\in\mathbb{Y}} d(s) < \frac{1}{4}$ and*

$$\mathrm{E}\,\|\varepsilon_0\|^4_{L^2(\mathbb{Y})} < \infty. \tag{3.9}$$

Then,

$$\sqrt{N}\begin{pmatrix}\widehat{\gamma}_{N,0} - \gamma_0\\ \vdots \\ \widehat{\gamma}_{N,H} - \gamma_H\end{pmatrix} \xrightarrow{d} G \doteq \begin{pmatrix} G_0 \\ \vdots \\ G_H\end{pmatrix} \in \left(\mathrm{HS}(L^2(\mathbb{Y})\right)^{\times(H+1)}, \tag{3.10}$$

where the convergence holds in the topology of $(\mathrm{HS}(L^2(\mathbb{Y})))^{\times(H+1)}$, G is a centered Gaussian element, and the covariance operator of G is the operator Σ given in (3.3)–(3.6), for $p, q = 0, \dots, H$, and with $\mathbb{H}$ replaced by $L^2(\mathbb{Y})$.

Proof: Let $\operatorname{ess\,sup}_{s\in\mathbb{Y}} d(s) = \frac{1}{4} - \delta$, for some $\delta \in (0, \frac{1}{4})$. We see that

$$\sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}}^{4/3} = \sum_{j=0}^{\infty}\left((j+1)^{\operatorname{ess\,sup}_{s\in\mathbb{Y}} d(s)-1}\right)^{4/3} = \sum_{j=0}^{\infty}(j+1)^{-1-\frac{4\delta}{3}} < \infty.$$

Then, Corollary 3.2 follows from Theorem 3.1. □

The second regime is more challenging. We first consider fluctuations for the autocovariance operators defined in (2.12) for X_n taking values in $\mathbb{H} = L^2(\mathbb{Y}, \mu)$ (Theorem 3.3). Then, we leverage this result and the spectral theorem for self-adjoint operators to identify the fluctuations of the autocovariance operator defined in (1.2) (Theorem 3.5).

In the second regime, the limit is no longer Gaussian. Instead, the resulting limit process can be represented as a double Wiener-Itô integral with sample paths in $L^2(\mathbb{Y}^2)$, applied to a certain kernel $f \in L^2(\mathbb{Y}^2)$ defined in (4.14) below. This limiting object is a generalization of the Rosenblatt distribution, allowing for a continuum of memory parameters, and reflecting the fact that the underlying process is infinite-dimensional. Double Wiener-Itô integrals with values in function spaces are introduced and defined in Section 4 below.

We define the suitable scaling as a multiplication operator on $L^2(\mathbb{Y}^2 : \mathbb{R}^{H+1})$ by

$$\Xi_N[f](r,s) \doteq N^{1-d(r)-d(s)} f(r,s), \quad f \in L^2(\mathbb{Y}^2 : \mathbb{R}^{H+1}), \quad r, s \in \mathbb{Y}. \tag{3.11}$$

Theorem 3.3. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be an $L^2(\mathbb{Y})$-valued linear process (2.8) and consider its autocovariance operators $(\widehat{\gamma}_{N,h}, h = 0, \dots, H)$ given in (2.12). Suppose $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right)$,*

$$\mathrm{E}\,\|\varepsilon_0\|^4_{L^4(\mathbb{Y})} < \infty. \tag{3.12}$$

and

$$\int_{\mathbb{Y}} \frac{\sigma^2(r)}{(1-2d(r))(2d(r)-1/2)}\mu(dr) < \infty. \tag{3.13}$$

Then,

$$\Xi_N\begin{pmatrix}\widehat{\gamma}_{N,0} - \gamma_0\\ \vdots \\ \widehat{\gamma}_{N,H} - \gamma_H\end{pmatrix} \xrightarrow{\mathrm{d}} \begin{pmatrix}\mathfrak{R} \\ \vdots \\ \mathfrak{R}\end{pmatrix} \in L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)}),$$

where $\mathfrak{R}$ is given in (4.14) below and Ξ_N in (3.11).

We continue with some remarks on the above assumptions.

Remark 3.4. 1. *Since $(\mathbb{Y}, \mu)$ is not necessarily a finite measure space, the condition* $\mathrm{E}\,\|\varepsilon_0\|^4_{L^4(\mathbb{Y})} < \infty$ *does not imply* $\mathrm{E}\,\|\varepsilon_0\|^2_{L^2(\mathbb{Y})} < \infty$*. However* $\mathrm{E}\,\|\varepsilon_0\|^2_{L^2(\mathbb{Y})} < \infty$ *is contained within Condition* (3.13)*.*

2. *If there exists some $\delta > 0$ such that $1/4 + \delta < d(r) < \frac{1}{2} - \delta$ for all $r \in \mathbb{Y}$, then the Conditions* $\mathrm{E}\,\|\varepsilon_0\|^2_{L^2(\mathbb{Y})} < \infty$ *and* (3.13) *are equivalent.*

3. *The estimates in* (3.12)*–*(3.13) *ensure that* (2.10) *is satisfied, hence the process $\{X_n\}_{n\in\mathbb{Z}}$ has a.s. sample paths in $L^2(\mathbb{Y})$; see Section 2.3. Moreover, the conditions in* (2.11) *are also satisfied due to* (3.12)*–*(3.13) *and since $d(r) > \frac{1}{4}$. Therefore, following upon our discussion in Section 2.3, the conditions of Theorem 3.3 imply that the sample mean of $\{X_n\}_{n\in\mathbb{Z}}$ satisfies a CLT.*

We turn to our final main result, which lifts the second regime to linear processes taking values in general Hilbert spaces. We start by introducing the right scaling. Define, for some unitary operator $U : \mathbb{H} \to L^2(\mathbb{Y})$, the operator $\Delta_N^U \in L\left((\mathbb{H}\otimes\mathbb{H})^{\times(H+1)}\right)$ by

$$\Delta_N^U[f] \doteq (U^* \otimes U^*)^{\times(H+1)}\, \Xi_N\, (U \otimes U)^{\times(H+1)}\, [f], \quad f \in (H\otimes H)^{\times(H+1)}, \tag{3.14}$$

where Ξ_N is the operator defined in (3.11). We then have the following theorem.

Theorem 3.5 (Second regime)**.** *Let $\{X_n\}_{n\in\mathbb{Z}}$ be an $\mathbb{H}$-valued linear process* (1.1)*,* (1.5)*, and consider its autocovariance operators $(\widehat{\Gamma}_{N,h}, h = 0,\dots,H)$ given in* (1.2)*. Suppose* (3.2)*,*

$$\mathrm{E}\,\|U\varepsilon_0\|^4_{L^4(\mathbb{Y})} < \infty \tag{3.15}$$

and

$$\int_{\mathbb{Y}} \frac{\mathrm{E}(U\varepsilon_0)^2(r)}{(1-2d(r))(2d(r)-1/2)}\mu(dr) < \infty. \tag{3.16}$$

Then,

$$\Delta_N^U \begin{pmatrix} \widehat{\Gamma}_{N,0} - \Gamma_0 \\ \vdots \\ \widehat{\Gamma}_{N,H} - \Gamma_H \end{pmatrix} \xrightarrow{\mathrm{d}} \begin{pmatrix} \mathfrak{Z}_U \\ \vdots \\ \mathfrak{Z}_U \end{pmatrix} \in (\mathbb{H}\otimes\mathbb{H})^{\times(H+1)},$$

where Δ_N^U is defined in (3.14) *and $\mathfrak{Z}_U$ is defined in* (4.14)*–*(4.15) *below.*

4 Double Wiener-Itô Integrals in Function Spaces

In this section, we construct double Wiener-Itô integrals with values in $L^2(\mathbb{Y}^2)$, extending the work of Norvaiša (1994) in the direction of integrators with spatial dependence, and the work of Fox and Taqqu (1987) in the direction of double integrals with sample paths in an infinite dimensional Hilbert space. Our candidate Wiener-Itô integrals are defined by integrating with respect to a family of dependent measures $\{W^{(r)}\}_{r\in\mathbb{Y}}$. In order to define a double Wiener-Itô integral in $L^2(\mathbb{Y}^2)$, we first define the integral for special kernels, and then use an approximation of f in terms of such special kernels. Note that $L^2(\mathbb{Y}^2)$ is a complete, separable, σ-finite measure space. Our construction is applicable for multiple Wiener-Itô integrals of any order, but we restrict ourselves to double Wiener-Itô integrals for simplicity. We start by recalling some elementary notions of the usual, $\mathbb{R}$-valued double Wiener-Itô integrals.

We write $(\mathbb{R}, \mathcal{B}, \lambda)$ for the measure space with the usual Borel topology $\mathcal{B}$ and the Lebesgue measure λ. For every $\psi \in L^2(\mathbb{R}^2, \lambda^2)$, the (standard) double Wiener-Itô integral with respect to a Gaussian random measure G is defined by

$$I_2(\psi) \doteq \int_{\mathbb{R}^2} \psi(x_1, x_2)\, G(dx_1)G(dx_2).$$

A Wiener-Itô integral of $\psi \in L^2(\mathbb{R}^2, \lambda^2)$ with regard to dependent integrators (4.1) was constructed in Fox and Taqqu (1987). Let G_1, G_2 be two dependent Gaussian random measures satisfying

$$\mathrm{E}\, G_1(A)G_2(B) = \sigma_{G_1G_2}\lambda(A \cap B), \quad A, B \in \mathcal{B}, \tag{4.1}$$

for some $\sigma_{G_1G_2} > 0$. Then, Fox and Taqqu (1987) introduced the corresponding double Wiener-Itô integral

$$\tilde{I}_2(\psi) = \int_{\mathbb{R}^2} \psi(x_1, x_2)\, G_1(dx_1)G_2(dx_2), \quad \psi \in L^2(\mathbb{R}^2). \tag{4.2}$$

In contrast to Fox and Taqqu (1987), we fix a correlation function $\sigma : \mathbb{Y}^2 \to \mathbb{R}$, denote $\sigma^2(r) \doteq \sigma(r, r)$, and consider a family of dependent Gaussian random measures $\{W^{(r)}(A) : A \in \mathcal{B}_0\}_{r \in \mathbb{Y}}$, where $\mathcal{B}_0 = \{A \in \mathcal{B} : \nu^{(r)}(A) = \sigma^2(r)\lambda(A) < \infty, \text{for all } r \in \mathbb{Y}\}$. All measures are defined on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let their covariances be given by, for $r, s \in \mathbb{Y}$, and $x \neq y$,

$$\mathrm{E}\, W^{(r)}(dx) = 0, \;\; \mathrm{E}\, W^{(r)}(dx)W^{(s)}(dx) = \sigma(r, s)dx, \;\; \mathrm{E}\, W^{(r)}(dx)W^{(s)}(dy) = 0. \tag{4.3}$$

Note that, for fixed r, the measure $W^{(r)}(dx)$ is viewed as the increment of an infinitesimal interval dx of a Brownian motion with coefficient $\sigma(r) \doteq \sqrt{\sigma^2(r)}$. For fixed $r, s \in \mathbb{Y}$, we set

$$\tilde{I}_2^{(r,s)}(\psi) = \int_{\mathbb{R}^2} \psi(x_1, x_2)\, W^{(r)}(dx_1)W^{(s)}(dx_2), \quad \psi \in L^2(\mathbb{R}^2), \tag{4.4}$$

where $\tilde{I}_2$ is the stochastic integral defined in (4.1)–(4.2), with $G_1 = W^{(r)}, G_2 = W^{(s)}$.

We aim to integrate with respect to the family of measures $\{W^{(r)}(A) : A \in \mathcal{B}_0\}_{r \in \mathbb{Y}}$. Therefore, we define the set of admissible kernels for such integrals. We call a map $f^{(r,s)}(x, y) \doteq f(r, s, x, y)$, $(r, s, x, y) \in \mathbb{Y}^2 \times \mathbb{R}^2$ an *admissible kernel* if (i) $f^{(r,s)}(x, y)$ is a $\mu^{\otimes 2} \otimes \lambda^{\otimes 2}$-measurable function, (ii) the family $\{\tilde{I}_2^{(r,s)}(f^{(r,s)}) : r, s \in \mathbb{Y}\}$ viewed as a stochastic process is measurable, where for fixed $r, s \in \mathbb{Y}$, $\tilde{I}_2^{(r,s)}(f^{(r,s)})$ is defined as in (4.3)–(4.4), and (iii) the map from $\mathbb{Y}^2 \times \mathbb{R}^2$ to $\mathbb{R}$ defined by

$$(r, s, x, y) \mapsto f^{\sigma}(r, s, x, y) \doteq f^{(r,s)}(x, y)\sigma(r)\sigma(s) \tag{4.5}$$

is in $L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))$ (with a slight abuse of notation). We denote by $\mathcal{H}^2 \doteq \mathcal{H}^2(\mathbb{Y}^2 \times \mathbb{R}^2)$ the space of admissible kernels satisfying Conditions (i)–(iii). Note that the space $\mathcal{H}^2$ depends on the selection of the function σ, but we suppress this dependence for notational simplicity. We present some remarks about the space $\mathcal{H}^2$.

Remark 4.1. 1. *If $\mathbb{Y}$ is a separable metric space equipped with its Borel σ-algebra and if the kernel f is a jointly measurable function on $\mathbb{Y}^2 \times \mathbb{R}^2$, then $\{\tilde{I}_2^{(r,s)} : r, s \in \mathbb{Y}\}$ has a measurable modification and can hence be assumed to be measurable; see Norvaiša (1994), p. 338.*

2. *In particular, Condition (iii) implies that $f^{(r,s)}(\cdot, \cdot) \in L^2(\mathbb{R}^2, \lambda^{\otimes 2})$, for μ-a.e. $r, s \in \mathbb{Y}$ since $\sigma^2(r) > 0$ for all $r \in \mathbb{Y}$.*

We proceed with defining our double Wiener-Itô integral for *special kernels*. We call an admissible kernel $f \in \mathcal{H}^2$ special, if there exists an $N \in \mathbb{N}$ and a system $\{\Delta_1, \dots, \Delta_N\}$ of disjoint sets in $\mathcal{B}_0$ such that, for $x, y \in \mathbb{R}$, $r, s \in \mathbb{Y}$,

$$f^{(r,s)}(x,y) = \begin{cases} c_{i_1,i_2}(r,s)\mathbb{1}_{\Delta_{i_1}}(x)\mathbb{1}_{\Delta_{i_2}}(y) & \text{for some } i_1, i_2 \in \{1, \dots, N\} \text{ such that } i_1 \neq i_2, \\ 0 & \text{else,} \end{cases} \tag{4.6}$$

where $c_{i_1,i_2} \in L^2(\mathbb{Y}^2, \mu^{\otimes 2})$ for all $(i_1, i_2) \in \{1, \dots, N\}^{\times 2}$. Then, the special admissible kernel $f^{(\cdot,\cdot)}(x,y)$ takes values in $L^2(\mathbb{Y}^2, \mu^{\otimes 2})$, and we can define the $L^2(\mathbb{Y}^2, \mu^{\otimes 2})$-valued stochastic process $\mathcal{I}_2(f)$ by

$$\mathcal{I}_2(f)(r,s) \doteq \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{N} c_{i_1,i_2}(r,s) W^{(r)}(\Delta_{i_1}) W^{(s)}(\Delta_{i_2}), \quad r, s \in \mathbb{Y}.$$

Definition 4.2. *Let $(\mathbb{Y}, \mu)$ be a σ-finite measure space, and let $\{W^{(r)}\}_{r \in \mathbb{Y}}$ be a family of Gaussian random measures on $(\mathbb{R}, \lambda)$ with covariances (4.3). Consider a kernel $f : \mathbb{Y}^2 \times \mathbb{R}^2 \to \mathbb{R}$. We say that there exists an $L^2(\mathbb{Y}^2)$-valued double Wiener-Itô integral of the kernel f, denoted $\mathcal{I}_2(f)$, if there exists a sequence $\{f_{(n)} : n \geqslant 1\}$ of $L^2(\mathbb{Y}^2)$-valued special kernels, defined on $\mathbb{Y}^2 \times \mathbb{R}^2$, such that*

(i) $\lim_{n \to \infty} \|f^{(r,s)} - f^{(r,s)}_{(n)}\|_{L^2(\mathbb{R}^2)} = 0$ $\mu^{\otimes 2}$*-a.s.,*

(ii) the sequence $\{\mathcal{I}_2(f_{(n)}) : n \geqslant 1\}$ is a Cauchy sequence in $L^1(\Omega : L^2(\mathbb{Y}^2))$.

In that case, we define the double Wiener-Itô integral by

$$\mathcal{I}_2(f) \doteq \lim_{n \to \infty} \mathcal{I}_2(f_{(n)}),$$

where the limit is again understood in $L^1(\Omega : L^2(\mathbb{Y}^2))$. For an $\mathbb{R}^d$−valued kernel $f = (f_1, \dots, f_d)$ such that $\mathcal{I}_2(f_i)$ exists for all $i = 1, \dots, d$, we fix the notation $\mathcal{I}_2(f) \doteq (\mathcal{I}_2(f_1), \dots, \mathcal{I}_2(f_d))$.

Whenever the underlying covariance structure of $\{W^{(r)}\}_{r \in \mathbb{Y}}$ is clear from the context, we write $\mathcal{I}_2$; otherwise, we write $\mathcal{I}_2^g$ to emphasize that the dependence of the underlying Gaussian random measure is characterized through (4.3).

Remark 4.3. *It follows that if there exists an $L^2(\mathbb{Y}^2)$-valued double Wiener-Itô integral $\mathcal{I}_2(f)$ of a kernel f, then the double Wiener-Itô integral $\mathcal{I}_2(f)$ has* P*-a.s. sample paths in $L^2(\mathbb{Y}^2)$ and is well defined, i.e., the definition of $\mathcal{I}_2(f)$ does not depend on the choice of a sequence $\{f_{(n)} : n \geqslant 1\}$ of $L^2(\mathbb{Y}^2)$-valued special functions. Moreover, the double Wiener-Itô integral $\mathcal{I}_2$ coincides with the one defined in Fox and Taqqu (1987), i.e.,*

$$\tilde{I}_2^{(r,s)}(f^{(r,s)}) = \mathcal{I}_2(f)(r,s), \quad \mu^{\otimes 2}\text{-a.s.,} \tag{4.7}$$

where $\tilde{I}_2$ is the stochastic integral defined in (4.3)–(4.4).

We show that the $L^2(\mathbb{Y}^2)$-valued double Wiener-Itô integral we defined exists for all kernels $f \in \mathcal{H}^2$. Note that the abstract machinery employed by Norvaiša (1994) to define such multiple stochastic integrals with values in functional spaces is not readily available. The analysis in Norvaiša (1994) relies heavily on a hypercontractivity property of the double stochastic integrals. However, we are not aware of whether the hypercontractivity property remains valid when the integration is conducted with regard to Gaussian measures with spatial dependence.

Theorem 4.4. *Let $f \in \mathcal{H}^2$. Then, the $L^2(\mathbb{Y}^2)$-valued double Wiener-Itô integral $\mathcal{I}_2(f)$ of the kernel f exists and*

$$\mathrm{E}\left(\int_{\mathbb{Y}^2} |\mathcal{I}_2(f^{(r,s)})|^2 \mu(dr)\mu(ds)\right)^{1/2} < \infty. \tag{4.8}$$

Moreover, the sequence $\{f_{(n)}\}_{n\in\mathbb{N}}$ approximating $f \in \mathcal{H}^2(\mathbb{Y}^2 \times \mathbb{R}^2)$ can be selected such that, for all $n \in \mathbb{N}$,

$$f_{(n)}^{(r,s)}(x,y) \leqslant f^{(r,s)}(x,y), \quad \textit{for all } (x,y) \in \mathbb{R}^2, (r,s) \in \mathbb{Y}^2. \tag{4.9}$$

Proof: We must show that there exists a sequence $f_{(n)}^{(r,s)}$ such that the definition requirements are satisfied for all $r, s \in \mathbb{Y}$. We start with item (i) of Definition 4.2.

For $r, s \in \mathbb{Y}$ fixed, consider the map $(x,y) \mapsto f^{(r,s)}(x,y)$ as an element of $L^2(\mathbb{R}^2)$. The set of special kernels is dense in $L^2(\mathbb{R})$ (for all r,s), and so there exists a family of functions $\{f_{(n)}^{(r,s)}\}_{n\in\mathbb{N},r,s\in\mathbb{Y}}$ such that the first item (i) of Definition 4.2 is true.

Moreover, we see that the simple functions can be chosen such that (4.9) is true as follows. We define a new sequence $\{\tilde{f}_{(n)}\}_{n\in\mathbb{N}}$ by

$$\tilde{f}_{(n)}^{(r,s)}(x,y) \doteq \begin{cases} f_{(n)}^{(r,s)}(x,y) & \text{if } f_{(n)}^{(r,s)}(x,y) \leqslant f^{(r,s)}(x,y), \\ f^{(r,s)}(x,y) - (f_{(n)}^{(r,s)}(x,y) - f^{(r,s)}(x,y)) & \text{if } f_{(n)}^{(r,s)}(x,y) > f^{(r,s)}(x,y). \end{cases}$$

Then, $f_{(n)}^{(r,s)}(x,y) \leqslant f^{(r,s)}(x,y)$ for all $x, y \in \mathbb{R}, r, s \in \mathbb{Y}$. Moreover, for fixed $r, s \in \mathbb{Y}$,

$$\begin{aligned} \|f^{(r,s)} - \tilde{f}_{(n)}^{(r,s)}\|_{L^2(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} |f^{(r,s)}(x,y) - \tilde{f}_{(n)}^{(r,s)}(x,y)|^2 dxdy \\ &= \int_{\mathbb{R}^2} |f^{(r,s)}(x,y) - f_{(n)}^{(r,s)}(x,y)|^2 dxdy \to 0, \end{aligned}$$

as $n \to \infty$.

We now turn to item (ii) of Definition 4.2. We show that $\{\mathcal{I}_2(f_{(n)})\}_{n\in\mathbb{N}}$ is Cauchy in the complete metric space $L^1(\Omega : L^2(\mathbb{Y}^2))$. In view of Remark 4.3, we write for $m, n \in \mathbb{N}$ and special kernels $f_{(m)}, f_{(n)}$,

$$\begin{aligned} &\|\mathcal{I}_2(f_{(m)}) - \mathcal{I}_2(f_{(n)})\|_{L^1(\Omega:L^2(\mathbb{Y}^2))} \\ &= \mathrm{E}\left|\int_{\mathbb{Y}^2} \left|\tilde{I}_2^{(r,s)}(f_{(n)}^{(r,s)}) - \tilde{I}_2^{(r,s)}(f_{(m)}^{(r,s)})\right|^2 \mu(dr)\mu(ds)\right|^{1/2} \\ &\leqslant \left|\mathrm{E}\int_{\mathbb{Y}^2} \left|\tilde{I}_2^{(r,s)}(f_{(n)}^{(r,s)}) - \tilde{I}_2^{(r,s)}(f_{(m)}^{(r,s)})\right|^2 \mu(dr)\mu(ds)\right|^{1/2} \\ &\leqslant \Bigg(\int_{\mathbb{Y}^2}\int_{\mathbb{R}^4} \left(f_{(m)}^{(r,s)}(x_1,x_2) - f_{(n)}^{(r,s)}(x_1,x_2)\right)\left(f_{(m)}^{(r,s)}(y_1,y_2) - f_{(n)}^{(r,s)}(y_1,y_2)\right) \\ &\qquad \times \mathrm{E}(W^{(r)}(dx_1)W^{(s)}(dx_2)W^{(r)}(dy_1)W^{(s)}(dy_2))\mu(dr)\mu(ds)\Bigg)^{1/2}, \end{aligned} \tag{4.10}$$

where the third line follows from Jensen's inequality by exchanging expectation and the square root operation. Recall that in this relation, with the integration over $\mathbb{R}^4$ we are excluding the diagonals $x_1 = x_2, y_1 = y_2$. From the calculation in (4.10), we have that

$$\begin{aligned} &\|\mathcal{I}_2(f_{(m)}^{(r,s)}) - \mathcal{I}_2(f_{(n)}^{(r,s)})\|_{L^1(\Omega:L^2(\mathbb{Y}^2))}^2 \\ &\leqslant \int_{\mathbb{Y}^2}\int_{\mathbb{R}^2} (f_{(m)}^{(r,s)}(x_1,x_2) - f_{(n)}^{(r,s)}(x_1,x_2))^2 \sigma^2(r)\sigma^2(s) dx_1 dx_2 \mu(dr)\mu(ds) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbb{Y}^2}\int_{\mathbb{R}^2} \left(f_{(m)}^{(r,s)}(x_1,x_2) - f_{(n)}^{(r,s)}(x_1,x_2)\right)\left(f_{(m)}^{(r,s)}(x_2,x_1) - f_{(n)}^{(r,s)}(x_2,x_1)\right) \\
&\times \sigma^2(r,s)dx_1dx_2\mu(dr)\mu(ds) \\
\leqslant\ & 4\int_{\mathbb{Y}^2}\int_{\mathbb{R}^2}(f_{(m)}^{(r,s)}(x_1,x_2) - f_{(n)}^{(r,s)}(x_1,x_2))^2\sigma^2(r)\sigma^2(s)dx_1dx_2\mu(dr)\mu(ds),
\end{aligned}
$$

where the last line follows from two iterations of Cauchy-Schwarz. Moreover, note that

$$
\begin{aligned}
&\int_{\mathbb{Y}^2}\int_{\mathbb{R}^2}(f_{(m)}(x_1,x_2) - f_{(n)}(x_1,x_2))^2\sigma^2(r)\sigma^2(s)dx_1dx_2\mu(dr)\mu(ds) \\
&\quad \leqslant 4\int_{\mathbb{Y}^2}\int_{\mathbb{R}^2}(f^{(r,s)}(x_1,x_2))^2\sigma^2(r)\sigma^2(s)dx_1dx_2\mu(dr)\mu(ds) = 4\|f^\sigma\|_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))} < \infty,
\end{aligned} \tag{4.11}
$$

since $f \in \mathcal{H}^2$ and with f^σ defined in (4.5). By DCT, it follows that we can select $n, m \in \mathbb{N}$ large enough such that

$$
\|\mathcal{I}_2(f_{(m)}) - \mathcal{I}_2(f_{(n)})\|^2_{L^1(\Omega:L^2(\mathbb{Y}^2))} \leqslant \varepsilon. \tag{4.12}
$$

The same argument shows that, for general $f \in \mathcal{H}^2$

$$
\mathrm{E}\left(\int_{\mathbb{Y}^2}|\mathcal{I}_2(f^{(r,s)})|^2\mu(dr)\mu(ds)\right)^{1/2} < \infty, \tag{4.13}
$$

which concludes the proof. □

4.1 Rosenblatt distribution with sample paths in $L^2(\mathbb{Y}^2)$

We define, for $d(r) \in \left(\frac{1}{4},\frac{1}{2}\right), r \in \mathbb{Y}$,

$$
\mathfrak{f}^{(r,s)}(x_1,x_2) := \int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv, \quad \mathfrak{R} \doteq \mathcal{I}_2(\mathfrak{f}), \tag{4.14}
$$

where $x_+ \doteq \max\{0,x\}$. We say that $\mathfrak{R}$ follows the *Rosenblatt distribution with sample paths in* $L^2(\mathbb{Y}^2)$. This aligns with the terminology presented in Veillette and Taqqu (2013), who consider the Rosenblatt distribution in the real-valued case.

To ensure that $\mathcal{I}_2(\mathfrak{f})$ is well-defined, we require the following result.

Lemma 4.5. *Let $d(r) \in \left(\frac{1}{4},\frac{1}{2}\right)$ for all $r \in \mathbb{Y}$. Then, the kernels $\mathfrak{f}$ defined in (4.14) belong to $\mathcal{H}^2$.*

Proof: We must check the three conditions for a kernel to be admissible. The first condition, i.e., measurability, can be seen from the form of the kernels $\mathfrak{f}^{(r,s)}$. The second condition is verified since $\mathbb{Y}^2$ is a separable metric space in the Borel topology; see Remark 4.1(1). The last condition follows from Lemma 6.2 below. □

4.2 Rosenblatt distribution with sample paths in general $\mathbb{H}\otimes\mathbb{H}$

We say that the $\mathbb{H}\otimes\mathbb{H}$-valued random variable $\mathfrak{Z}_U$ defined as

$$
\mathfrak{Z}_U \doteq (U^*\otimes U^*)\mathcal{I}_2^{\sigma_U}(\mathfrak{f}), \tag{4.15}
$$

follows a Rosenblatt distribution when $\mathfrak{f}$ is as in (4.14), U is a unitary operator as in the spectral decomposition (2.6), and σ_U is defined by

$$
\sigma_U(r,s) \doteq \mathrm{E}\left(((U\varepsilon)(r))((U\varepsilon)(s))\right), \quad r,s \in \mathbb{Y}.
$$

Note that in (4.15), we slightly abused notation, by viewing the sample paths of $\mathcal{I}_2^{\sigma_U}(\mathfrak{f})$ in the isomorphic space $L^2(\mathbb{Y}^2) \cong L^2(\mathbb{Y})\otimes L^2(\mathbb{Y})$.

5 Proofs of Main Results

In this section, we give the proofs of our main results. Theorems 3.1, 3.3, and 3.5 are respectively proven in Sections 5.1, 5.2, and 5.3.

5.1 Proof of Theorem 3.1

The result can be inferred by following the proof of Theorem 5 in Mas (2002). We show that the arguments in Mas (2002) remain true after replacing Assumption H.2 of Mas (2002) by the weaker assumption (3.1). We note that Mas (2002) considers two-sided (non-causal) series instead of one sided (causal) series, but this does not affect the analysis.

To complete the proof, we must show that Theorem 5 of Mas (2002) holds under the weaker assumption (3.1). In turn, it suffices to prove that Lemma 8 in Mas (2002) holds, i.e., we must check that

(i) $\sum_{h=0}^{\infty} \|\Gamma_{h+p-q} T \Gamma_h\|_{\mathrm{Tr}} < \infty$,

(ii) $\sum_{h=0}^{\infty} \|\Gamma_{h+q} T \Gamma_{h-p}\|_{\mathrm{Tr}} < \infty$, and

(iii) $\sum_{h=0}^{\infty} \|u_h (\Lambda - \Phi) u_{h+q}^*\|_{\mathrm{Tr}} < \infty$,

where Λ, Φ, Γ were defined in (3.5)–(3.6). We start with the third estimate (iii). Note that

$$\begin{aligned} \sum_{h=0}^{\infty} \|u_h (\Lambda - \Phi) u_{h+q}^*\|_{\mathrm{Tr}} &\leqslant (\|\Lambda\|_{\mathrm{Tr}} + \|\Phi\|_{\mathrm{Tr}}) \sum_{h=0}^{\infty} \|u_h\|_{\mathrm{op}} \|u_{h+q}\|_{\mathrm{op}} \\ &\leqslant \left(\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 + 3\,\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 \right) \sum_{h=0}^{\infty} \|u_h\|_{\mathrm{op}}^2 < \infty. \end{aligned} \tag{5.1}$$

For the second line in (5.1) we used that, without loss of generality, we can select $\{u_j\}_{j\in\mathbb{N}_0}$ in the representation (1.1) such that $\{\|u_j\|_{\mathrm{op}}\}_{j\in\mathbb{N}_0}$ is decreasing. More precisely, we can define a renumeration $\{\tilde{u}_j\}_{j\in\mathbb{N}_0}$ of $\{u_j\}_{j\in\mathbb{N}_0}$ such that $\{\tilde{u}_j\}_{j\in\mathbb{N}_0}$ is decreasing in the operator norm. Then, take $\tilde{X}_n \doteq \sum_{j=0}^{\infty} \tilde{u}_j \varepsilon_{n-j}$. Note that, for all n, $X_n = \tilde{X}_n$ in law, since $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is i.i.d., and so we can work with $\tilde{X}_n$ instead of X_n (and the same for their respective autocovariance operators).

We now turn to the first estimate (i), and (ii) follows by identical calculations. By denoting $M \doteq \sum_{j=0}^{\infty} \|u_j\|_{\mathrm{op}}^{4/3} < \infty$ and since $\{\|u_j\|_{\mathrm{op}}\}_{j\in\mathbb{N}_0}$ is decreasing in j, we write

$$\begin{aligned} \sum_{h=0}^{\infty} \|\Gamma_{h+p-q} T \Gamma_h\|_{\mathrm{Tr}} &\leqslant \mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 \sum_{i=0}^{\infty} \sum_{j=p}^{\infty} \sum_{h=0}^{\infty} \|u_{h+i}\|_{\mathrm{op}} \|u_i\|_{\mathrm{op}} \|u_{h+q+j}\|_{\mathrm{op}} \|u_{p+j}\|_{\mathrm{op}} \\ &\leqslant \mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 \sum_j \sum_h \|u_{h+q+j}\|_{\mathrm{op}} \|u_{p+j}\|_{\mathrm{op}} \|u_h\|_{\mathrm{op}}^{2/3} \sum_i \|u_{h+i}\|_{\mathrm{op}}^{1/3} \|u_i\|_{\mathrm{op}} \\ &\leqslant M\,\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 \sum_h \|u_h\|_{\mathrm{op}}^{2/3} \|u_{h+q+j}\|_{\mathrm{op}}^{2/3} \sum_j \|u_{h+q+j}\|_{\mathrm{op}}^{1/3} \|u_{p+j}\|_{\mathrm{op}} \\ &\leqslant M^2\,\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4 \sum_h \|u_h\|_{\mathrm{op}}^{2/3} \|u_{h+q+j}\|_{\mathrm{op}}^{2/3} \\ &\leqslant M^3\,\mathrm{E}\,\|\varepsilon_0\|_{\mathbb{H}}^4. \end{aligned} \tag{5.2}$$

The estimate in the first line follows from calculations in pp. 127-128 of Mas (2002). Coupled with the strategy followed there, these estimates finish the proof.

5.2 Proof of Theorem 3.3

In the proof of Theorem 3.3, we are concerned with the case $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right), s \in \mathbb{Y}$. The proof is structured as follows. We first separate the sample autocovariances into their diagonal and off-diagonal terms. Lemma 5.1 below shows that only the off-diagonal terms contribute to the limit in the sought convergence of the sample autocovariance operators. We then leverage Lemma 6.3 below to identify the scaling limit (in $L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$) of the off-diagonal terms. Lemma 6.3 is the key result in our analysis and its application gives the sought convergence result.

The sample autocovariances can be separated into diagonal and off-diagonal parts as follows. For $h = 0, \ldots, H$, we have

$$\begin{aligned}
\widehat{\gamma}_{N,h}(r,s) - \gamma_h(r,s) &= \frac{1}{N}\sum_{n=1}^{N} X_{n+h}(r)X_n(s) - \sum_{j=0}^{\infty}(j+h+1)^{d(r)-1}(j+1)^{d(s)-1}\sigma(r,s) \\
&= \frac{1}{N}\sum_{n=1}^{N}\sum_{j=0}^{\infty} u_{j+h}(r)u_j(s)(\varepsilon_{n-j}(r)\varepsilon_{n-j}(s) - \sigma(r,s)) \\
&\quad + \frac{1}{N}\sum_{n=1}^{N}\sum_{j\neq i+h} u_j(r)u_i(s)\varepsilon_{n+h-j}(r)\varepsilon_{n-i}(s) \\
&= D_{N,h}(r,s) + O_{N,h}(r,s)
\end{aligned} \tag{5.3}$$

with $u_j(r) = (j+1)^{d(r)-1}$ as in (2.8) and

$$\begin{aligned}
D_{N,h}(r,s) &\doteq \frac{1}{N}\sum_{n=1}^{N}\sum_{j=0}^{\infty} u_{j+h}(r)u_j(s)(\varepsilon_{n-j}(r)\varepsilon_{n-j}(s) - \sigma(r,s)), \\
O_{N,h}(r,s) &\doteq \frac{1}{N}\sum_{n=1}^{N}\sum_{\substack{i,j=0,\ldots,\infty \\ j\neq i+h}} u_j(r)u_i(s)\varepsilon_{n+h-j}(r)\varepsilon_{n-i}(s).
\end{aligned} \tag{5.4}$$

By Theorem 2.3 in Bosq (2000), it suffices to show

$$\|\Xi_N(D_{N,h}, h = 0, \ldots, H)\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \xrightarrow{\mathrm{P}} 0, \tag{5.5}$$

$$\Xi_N(O_{N,h}, h = 0, \ldots, H) \xrightarrow{\mathrm{d}} (\mathfrak{R}, h = 0, \ldots, H). \tag{5.6}$$

The convergences (5.5) and (5.6) follow by Lemmas 5.1 and 5.3 respectively.

Lemma 5.1. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be as in Theorem 3.3, with $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right)$ for $\mu-a.s.$ $s \in \mathbb{Y}$. Then, (5.5) holds.*

Proof: By Markov's inequality, for all $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{P}\left(\|\Xi_N(D_{N,h}, h = 0, \ldots, H)\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} > \varepsilon\right) &\leqslant \frac{1}{\varepsilon^2}\,\mathrm{E}\,\|\Xi_N(D_{N,h}, h = 0, \ldots, H)\|^2_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \\
&= \frac{1}{\varepsilon^2}\int\int \mathrm{E}\,\|N^{1-d(r,s)}(D_{N,h}(r,s), h = 0, \ldots, H)\|^2_{\mathbb{R}^{H+1}}\mu(dr)\mu(ds) \\
&= \frac{1}{\varepsilon^2}\sum_{h=0}^{H}\int\int \mathrm{E}\,|N^{1-d(r,s)}D_{N,h}(r,s)|^2\mu(dr)\mu(ds).
\end{aligned}$$

It now suffices to show that the last term converges to 0 as $N \to \infty$. Recalling (5.4), note that we can rewrite the diagonal term as

$$D_{N,h} = \frac{1}{N}\sum_{n=1}^{N} Y_n \in L^2(\mathbb{Y}^2), \quad Y_n(r,s) \doteq \sum_{j=0}^{\infty} \alpha_j^{(h)}[\zeta_{j-n}](r,s), \tag{5.7}$$

where the sequences $\alpha_j^{(h)}, \zeta_j$ are respectively $L(L^2(\mathbb{Y}^2))$- and $L^2(\mathbb{Y}^2)$-valued elements given by

$$\alpha_j^{(h)}[f](r,s) \doteq u_{j+h}(r)u_j(s)f(r,s), \quad \zeta_j(r,s) \doteq \varepsilon_j(r)\varepsilon_j(s) - \sigma(r,s), \quad f \in L^2(\mathbb{Y}^2),\ r,s \in \mathbb{Y},\ j \geqslant 0. \tag{5.8}$$

Moreover, note that $\{\zeta_j\}_{j\in\mathbb{Z}}$ are centered and i.i.d. with

$$\mathrm{E}\left(\zeta_i(r,s)\zeta_j(r,s)\right) = \begin{cases} \mathrm{E}\,\varepsilon_0^2(r)\varepsilon_0^2(s) - (\sigma(r,s))^2 & i = j \\ 0 & i \neq j. \end{cases}, \quad \sigma_\zeta^2(r,s) \doteq \mathrm{E}\left(\zeta_0(r,s)\zeta_0(r,s)\right). \tag{5.9}$$

We claim that there exists a constant $c > 0$, such that

$$\begin{aligned} &\int\!\!\int \mathrm{E}\,|N^{1-d(r,s)} D_{N,h}(r,s)|^2 \mu(dr,ds) = \int N^{2-2d(r,s)}\,\mathrm{E}\,|D_{N,h}(r,s)|^2 \mu(dr,ds) \\ &\leqslant c \max\{\log^2(N)/2, \log(N)\} \int\!\!\int N^{1-2d(r,s)} |\sigma_\zeta(r,s)|^2 \mu(dr)\mu(ds), \end{aligned} \tag{5.10}$$

where σ_ζ was defined in (5.9). Then, the quantity in the second line of (5.10) converges to zero by the dominated convergence theorem and the observation that $\max\{\log^2(N)/2, \log(N)\} N^{1-2d(r,s)} \to 0$ as $N \to \infty$ for a.s. $r,s \in \mathbb{Y}$, since $\log(N), \log^2(N)$ are slowly varying functions and $2d(r,s) < 1$ for μ-a.s. $r,s \in \mathbb{Y}$.

To show the inequality in (5.10), first note that

$$\mathrm{E}\left(D_{N,h}(r,s)\right)^2 = \frac{1}{N}\,\mathrm{E}\left(Y_0(r,s)\right)^2 + \frac{2}{N^2}\sum_{n=1}^{N-1}\sum_{l=n+1}^{N} \mathrm{E}\left(Y_n(r,s)Y_l(r,s)\right). \tag{5.11}$$

We can see that

$$\begin{aligned} \mathrm{E}\left(Y_0(r,s)\right)^2 &= \sum_{i=0}^{\infty}(i+1)^{d(r,r)-2}(i+h+1)^{d(s,s)-2}\sigma_\zeta^2(r,s) \\ &\leqslant \sigma_\zeta^2(r,s)\sum_{i=0}^{\infty}(i+1)^{d(r,r)+d(s,s)-4} < \infty. \end{aligned} \tag{5.12}$$

Moreover, by using (5.9), it follows that

$$\begin{aligned} &\mathrm{E}\left(Y_0(r,s)Y_l(r,s)\right) \\ &= \mathrm{E}\left(\sum_{i,j=0}^{\infty} \alpha_i^{(h)}[\zeta_i](r,s)\alpha_j^{(h)}[\zeta_{j-l}](r,s)\right) \\ &= \sum_{i,j=0}^{\infty}(i+1)^{d(r)-1}(i+h+1)^{d(s)-1}(j+1)^{d(r)-1}(j+h+1)^{d(s)-1}\,\mathrm{E}\left(\zeta_i(r,s)\zeta_{j-l}(r,s)\right) \\ &= \sum_{j=0}^{\infty}(j+l+1)^{d(r)-1}(j+l+h+1)^{d(s)-1}(j+1)^{d(r)-1}(j+h+1)^{d(s)-1}\sigma_\zeta^2(r,s) \end{aligned}$$

$$\leqslant \sum_{j=0}^{\infty}(j+l+1)^{d(r,s)-2}(j+1)^{d(r,s)-2}|\sigma_{\zeta}(r,s)|^2 \leqslant |\sigma_{\zeta}(r,s)|^2\sum_{j=0}^{\infty}(j+l+1)^{-1}(j+1)^{-1}. \quad (5.13)$$

Moreover, by the proof of Proposition 1 in Characiejus and Račkauskas (2013),

$$\sum_{j=0}^{\infty}(j+l+1)^{-1}(j+1)^{-1} \leqslant (l+1)^{-1}+l^{-1}\left(\log\left(\frac{l+1}{2}\right)+\int_1^{\infty}(x(x+1))^{-1}dx\right) \leqslant cl^{-1}\log(l),$$

so that, there is a constant, $c>0$, with

$$\sum_{j=0}^{\infty}(j+l+1)^{-1}(j+1)^{-1} \leqslant cl^{-1}\log(l). \quad (5.14)$$

Then the last term in (5.11) can be estimated upon noticing that

$$\begin{aligned}
\frac{1}{N^2}\sum_{n=1}^{N-1}\sum_{l=n+1}^{N}\mathrm{E}\left(Y_n(r,s)Y_l(r,s)\right) &= \frac{1}{N^2}\sum_{l=1}^{N-1}(N-l)\,\mathrm{E}\left(Y_0(r,s)Y_l(r,s)\right)\\
&\leqslant c\frac{1}{N^2}\sum_{l=1}^{N-1}(N-l)l^{-1}\log(l)|\sigma_{\zeta}(r,s)|^2 \quad (5.15)\\
&\leqslant c\frac{1}{N}\sum_{l=1}^{N-1}(1-\frac{l}{N})l^{-1}\log(l)|\sigma_{\zeta}(r,s)|^2\\
&\leqslant c|\sigma_{\zeta}(r,s)|^2\frac{\log N}{N}\sum_{l=1}^{N-1}l^{-1}\\
&\leqslant c\frac{1}{N}\max\{\log^2(N)/2,\log(N)\}|\sigma_{\zeta}(r,s)|^2, \quad (5.16)
\end{aligned}$$

where (5.15) follows from (5.13)–(5.14). Then, (5.10) follows from (5.11), (5.12), and (5.16). □

Remark 5.2. *In the case* ess $\sup_{s\in\mathbb{Y}} d(s) < \frac{1}{2}$, $\{\alpha_j^{(h)}\}_{j\in\mathbb{N}}$ *in the proof of Lemma 5.1 are absolutely summable in the operator norm. Then, Lemma 5.1, as well as the asymptotic normality for* $D_{N,h}$*, follows from Theorem 2 in Merlevède et al. (1997). This argument is no longer available when* ess $\sup_{s\in\mathbb{Y}} d(s) = \frac{1}{2}$*, thus justifying the proof strategy pursued here.*

Lemma 5.3. *Let* $\{X_n\}_{n\in\mathbb{Z}}$ *be as in Theorem 3.3, with* $d(s)\in\left(\frac{1}{4},\frac{1}{2}\right)$ *for* $\mu-$*a.s.* $s\in\mathbb{Y}$*. Then,* (5.6) *holds.*

Proof: To investigate the asymptotic behavior of the off-diagonal terms in (5.6), we write

$$\begin{aligned}
N^{1-d(r,s)}O_{N,h}(r,s) &= N^{-d(r,s)}\sum_{n=1}^{N}\sum_{\substack{i,j=0\\ j\neq i+h}}^{\infty}u_j(r)u_i(s)\varepsilon_{n-(j-h)}(r)\varepsilon_{n-i}(s)\\
&= N^{-d(r,s)}\sum_{n=1}^{N}\sum_{\substack{j_1=-h,j_2=0\\ j_1\neq j_2}}^{\infty}u_{j_1+h}(r)u_{j_2}(s)\varepsilon_{n-j_1}(r)\varepsilon_{n-j_2}(s)\\
&= N^{-d(r,s)}\sum_{n=1}^{N}\sum_{\substack{j_1=-\infty,\dots,n+h\\ j_2=-\infty,\dots,n\\ j_1\neq j_2}}u_{n+h-j_1}(r)u_{n-j_2}(s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)
\end{aligned}$$

$$= \sum_{\substack{j_1,j_2=-\infty \\ j_1\neq j_2}}^{\infty} N^{-d(r,s)} \sum_{n=1}^{N} u_{n+h-j_1}(r)u_{n-j_2}(s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\mathbb{1}_{\{n+h-j_1\geqslant 0\}}\mathbb{1}_{\{n-j_2\geqslant 0\}}$$
$$= \sum_{j_1\neq j_2} C_{N,h}(j_1,j_2,r,s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s),$$

where we have used the changes of variables $j_1 = j - h$ ($j_2 = i$), and $j_k = n - j_k$, $k = 1, 2$, for the second and third lines respectively, and where

$$C_{N,h}(j_1,j_2,r,s) \doteq N^{-d(r,s)} \sum_{n=1}^{N} u_{n+h-j_1}(r)u_{n-j_2}(s)\mathbb{1}_{\{n+h-j_1\geqslant 0\}}\mathbb{1}_{\{n-j_2\geqslant 0\}}. \tag{5.17}$$

Then,

$$\Xi_N(O_{N,h}, h = 0,\dots,H)(r,s) = \sum_{j_1\neq j_2} C_N(j_1,j_2,r,s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s),\ r,s\in\mathbb{Y}, \tag{5.18}$$

with $C_N(j_1,j_2,\cdot,\cdot) \in L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$ defined by

$$C_N(j_1,j_2,r,s) = (C_{N,0}(j_1,j_2,r,s),\dots,C_{N,H}(j_1,j_2,r,s))'. \tag{5.19}$$

Moreover, define for $r,s\in\mathbb{Y}$, $h = 0,\dots,H$, and $x_1,x_2\in\mathbb{R}$,

$$\widetilde{C}_N(x_1,x_2,\cdot,\cdot) \doteq NC_N(\lceil x_1N\rceil,\lceil x_2N\rceil,\cdot,\cdot), \quad \widetilde{C}_N^{\sigma}(x_1,x_2,r,s) \doteq \widetilde{C}_N(x_1,x_2,r,s)\sigma(r)\sigma(s). \tag{5.20}$$

Then, from (5.17),

$$\widetilde{C}_{N,h}(x_1,x_2,r,s)$$
$$= N^{1-d(r,s)} \sum_{n=1}^{N} u_{n+h-\lceil x_1N\rceil}(r)u_{n-\lceil x_2N\rceil}(s)\mathbb{1}_{\{n+h-\lceil x_1N\rceil\geqslant 0\}}\mathbb{1}_{\{n-\lceil x_2N\rceil\geqslant 0\}}$$
$$= N^{2-d(r,s)} \int_0^1 u_{\lceil vN\rceil+h-\lceil x_1N\rceil}(r)u_{\lceil vN\rceil-\lceil x_2N\rceil}(s)\mathbb{1}_{\{\lceil vN\rceil+h-\lceil x_1N\rceil\geqslant 0\}}\mathbb{1}_{\{\lceil vN\rceil-\lceil x_2N\rceil\geqslant 0\}}dv. \tag{5.21}$$

We seek to apply Lemma 6.3 to identify the weak limits of $\Xi_N(O_{N,h}, h = 0,\dots,H)$. For this reason, we set

$$\bar{\mathfrak{f}}^{\sigma} \doteq (\mathfrak{f}^{\sigma}, h = 0,\dots,H)', \quad \mathfrak{f}^{\sigma}(x,y,r,s) \doteq \mathfrak{f}^{(r,s)}(x,y)\sigma(r)\sigma(s) \tag{5.22}$$

as in (4.5), where $\mathfrak{f}$ was defined in (4.14). Here we maintain the notation σ to remember the dependence on the correlation structure of the underlying noise. Then, the previous relation implies that,

$$\left\|\widetilde{C}_N^{\sigma} - \bar{\mathfrak{f}}^{\sigma}\right\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{Y}^2} \Bigg\| \Bigg(\int_0^1 N^{2-d(r,s)} u_{\lceil vN\rceil+h-\lceil x_1N\rceil}(r)u_{\lceil vN\rceil-\lceil x_2N\rceil}(s)\mathbb{1}_{\{\lceil vN\rceil+h-\lceil x_1N\rceil\geqslant 0\}}\mathbb{1}_{\{\lceil vN\rceil-\lceil x_2N\rceil\geqslant 0\}}$$
$$- (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv, h = 0,\dots,H \Bigg) \Bigg\|^2_{\mathbb{R}^{H+1}} \sigma^2(r)\sigma^2(s)drdsdx_1dx_2. \tag{5.23}$$

Note that to show $\left\|\widetilde{C}_N^{\sigma} - \bar{\mathfrak{f}}^{\sigma}\right\|^2 \to 0$, it remains to show that the integrand is (i) integrable, and (ii) converges pointwise (in r, s, x_1, x_2, as $N \to \infty$) to 0. Upon establishing these two claims, the DCT will ensure that the integral tends to zero.

For (i), recall that from Lemma 6.2 we have that $\mathfrak{f}^\sigma \in L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))$. For the sake of notational simplicity, we set $H = 0$ (and so $h = 0$), and we show that $\widetilde{C}_N^\sigma \in L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))$. The case $h \in \mathbb{N}$ follows from similar calculations. First, note that

$$\begin{aligned} &N^{1-d(r)} u_{\lceil vN\rceil - \lceil x_1 N\rceil}(r) \mathbb{1}_{\{\lceil vN\rceil - \lceil x_1 N\rceil \geqslant 0\}} \\ &\quad = \left(\frac{1}{N}\right)^{d(r)-1} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_1 N\rceil\}} + \left(\frac{\lceil vN\rceil}{N} - \frac{\lceil x_1 N\rceil}{N} + \frac{1}{N}\right)^{d(r)-1} \mathbb{1}_{\{\lceil vN\rceil - \lceil x_1 N\rceil \geqslant 1\}}, \end{aligned} \tag{5.24}$$

where it follows that, for all $x_1 \in \mathbb{R}, r \in \mathbb{Y}, N \in \mathbb{N}$,

$$\left(\frac{\lceil vN\rceil}{N} - \frac{\lceil x_1 N\rceil}{N} + \frac{1}{N}\right)^{d(r)-1} \mathbb{1}_{\{\lceil vN\rceil - \lceil x_1 N\rceil \geqslant 1\}} \leqslant (v - x_1)_+^{d(r)-1}, \tag{5.25}$$

by recalling that $v \leqslant \frac{\lceil vN\rceil}{N}$, $\frac{\lceil x_1 N\rceil}{N} \leqslant x_1 + \frac{1}{N}$, and that $d(r) - 1 < 0$. Analogous calculations show that

$$\begin{aligned} &N^{1-d(s)} u_{\lceil vN\rceil - \lceil x_2 N\rceil}(s) \mathbb{1}_{\{\lceil vN\rceil - \lceil x_2 N\rceil \geqslant 0\}} \\ &\quad = \left(\frac{1}{N}\right)^{d(s)-1} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_2 N\rceil\}} + \left(\frac{\lceil vN\rceil}{N} - \frac{\lceil x_2 N\rceil}{N} + \frac{1}{N}\right)^{d(s)-1} \mathbb{1}_{\{\lceil vN\rceil - \lceil x_1 N\rceil \geqslant 1\}} \\ &\quad \leqslant \left(\frac{1}{N}\right)^{d(s)-1} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_2 N\rceil\}} + (v - x_2)_+^{d(s)-1}. \end{aligned} \tag{5.26}$$

Combining (5.21), (5.24), (5.25), and (5.26), we see that

$$\left\| \widetilde{C}_N^\sigma \right\|^2_{L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))} \leqslant 4 \left(\mathcal{R}_{1,N} + \mathcal{R}_{2,N} + \mathcal{R}_{3,N} + \|\mathfrak{f}^\sigma\|^2_{L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))} \right),$$

where

$$\begin{aligned} \mathcal{R}_{1,N} &\doteq \left\| \sigma(r)\sigma(s) N^{2-d(r,s)} \int_0^1 \mathbb{1}_{\{\lceil vN\rceil = \lceil x_1 N\rceil\}} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_2 N\rceil\}} dv \right\|^2_{L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))}, \\ \mathcal{R}_{2,N} &\doteq \left\| \sigma(r)\sigma(s) N^{1-d(r)} \int_0^1 (v - x_2)_+^{d(s)-1} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_1 N\rceil\}} dv \right\|^2_{L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))}, \\ \mathcal{R}_{3,N} &\doteq \left\| \sigma(r)\sigma(s) N^{1-d(s)} \int_0^1 (v - x_1)_+^{d(r)-1} \mathbb{1}_{\{\lceil vN\rceil = \lceil x_2 N\rceil\}} dv \right\|^2_{L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))}. \end{aligned} \tag{5.27}$$

We now show that $\mathcal{R}_{i,N} \to 0$ as $N \to \infty$ for $i = 1, 2, 3$. We first investigate $\mathcal{R}_{1,N}$. Note that

$$\begin{aligned} \mathcal{R}_{1,N} &= \int_{\mathbb{Y}^2} \sigma^2(r)\sigma^2(s) N^{4-2d(r,s)} \int_{\mathbb{R}^2} \int_0^1 \int_0^1 \mathbb{1}_{\{\lceil v_1 N\rceil = \lceil x_1 N\rceil, \lceil v_1 N\rceil = \lceil x_2 N\rceil, \lceil v_2 N\rceil = \lceil x_1 N\rceil, \lceil v_2 N\rceil = \lceil x_2 N\rceil\}} \\ &\qquad\qquad \times dv_1 dv_2 dx_1 dx_2 \mu(dr)\mu(ds) \\ &= \int_{\mathbb{Y}^2} \sigma^2(r)\sigma^2(s) N^{1-2d(r,s)} \mu(dr)\mu(ds) \\ &= \left(\int_{\mathbb{Y}} \sigma^2(r) N^{1/2-2d(r)} \mu(dr) \right)^2, \end{aligned} \tag{5.28}$$

where the second equality follows upon noticing that

$$\int_{\mathbb{R}^2}\int_0^1\int_0^1 \mathbb{1}_{\{\lceil v_1N\rceil=\lceil x_1N\rceil,\lceil v_1N\rceil=\lceil x_2N\rceil,\lceil v_2N\rceil=\lceil x_1N\rceil,\lceil v_2N\rceil=\lceil x_2N\rceil\}}$$
$$= \sum_{i=0}^{N-1}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N} dv_1dv_2dx_1dx_2 = N^{-3}.$$

Since $d(r) > \frac{1}{4}$, it follows from (5.28) and DCT that $\mathcal{R}_{1,N} \to 0$. We now turn to proving $\mathcal{R}_{2,N} \to 0$ and the calculations are analogous (but symmetrical in r and s) for $\mathcal{R}_{3,N} \to 0$. First, observe that

$$\begin{aligned}&\left(N^{1-d(r)}\int_0^1 (v-x_2)_+^{d(s)-1}\mathbb{1}_{\{\lceil vN\rceil=\lceil x_1N\rceil\}}\right)^2\\ &\quad= N^{2-2d(r)}\int_0^1\int_0^1 (v_1-x_2)_+^{d(s)-1}(v_2-x_2)_+^{d(s)-1}\mathbb{1}_{\{\lceil v_1N\rceil=\lceil x_1N\rceil\}}\mathbb{1}_{\{\lceil v_2N\rceil=\lceil x_1N\rceil\}}dv_1dv_2,\end{aligned} \tag{5.29}$$

and moreover

$$\begin{aligned}&\int_{\mathbb{R}}\mathbb{1}_{\{\lceil v_1N\rceil=\lceil x_1N\rceil\}}\mathbb{1}_{\{\lceil v_2N\rceil=\lceil x_1N\rceil\}}dx_1\\ &\leqslant \mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\sum_{i=0}^{N-1}\int_{i/N}^{(i+1)/N}\mathbb{1}_{\{v_1\in\left(\frac{i}{N},\frac{i+1}{N}\right)\}}dx_1 = \mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\frac{1}{N}.\end{aligned} \tag{5.30}$$

In addition, we see that

$$\begin{aligned}&\int_0^1\int_0^1 \mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_{\mathbb{R}}(v_1-x_2)_+^{d(s)-1}(v_2-x_2)_+^{d(s)-1}dx_2dv_2dv_1\\ &=\int_0^1\int_0^1 \mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_{-\min\{v_1,v_2\}}^{\infty}(v_1+x_2)_+^{d(s)-1}(v_2+x_2)_+^{d(s)-1}dx_2dv_2dv_1\\ &=\int_0^1\Bigg[\int_0^{v_1}\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_{-v_2}^{\infty}(v_1+x_2)^{d(s)-1}(v_2+x_2)^{d(s)-1}dx_2dv_2\\ &\qquad+\int_{v_1}^1\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_{-v_1}^{\infty}(v_1+x_2)^{d(s)-1}(v_2+x_2)^{d(s)-1}dx_2dv_2\Bigg]dv_1\\ &=\int_0^1\Bigg[\int_0^{v_1}\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_0^{\infty}z^{d(s)-1}(z+(v_1-v_2))^{d(s)-1}dzdv_2\\ &\qquad+\int_{v_1}^1\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_0^{\infty}z^{d(s)-1}(z+(v_2-v_1))^{d(s)-1}dx_2dv_2\Bigg]dv_1\\ &=\int_0^1\int_0^1\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}\int_0^{\infty}z^{d(s)-1}(z+|v_1-v_2|)^{d(s)-1}dzdv_2dv_1\\ &=\int_0^1\int_0^1\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}|v_2-v_1|^{2d(s)-1}B(d(s),1-2d(s))dv_1dv_2\\ &\leqslant B(d(s),1-2d(s))N^{-2d(s)}.\end{aligned} \tag{5.31}$$

Here, the first equality followed from the change of variables $-x_2 \mapsto x_2$; the fifth and sixth lines follow by the change of variables $x_2 + v_2 = z$ and $x_2 + v_1 = z$ respectively; finally the last line follows upon noticing that, on the event $\mathbb{1}_{\{\lceil v_1N\rceil=\lceil v_2N\rceil\}}$, we have $|v_2 - v_1| \leqslant N^{-1}$.

By combining (5.29), (5.30), and (5.31) we have that

$$
\begin{aligned}
\mathcal{R}_{2,N} \leqslant \int_{\mathbb{Y}^2} \sigma^2(r)\sigma^2(s)N^{1-2d(r)} \int_0^1 \int_0^1 \mathbb{1}_{\{\lceil v_1 N\rceil = \lceil v_2 N\rceil\}} \\
\times \int_{\mathbb{R}} (v_1 - x_2)_+^{d(s)-1}(v_2 - x_2)_+^{d(s)-1} dx_2 dv_1 dv_2 \mu(dr)\mu(ds) \\
\leqslant \int_{\mathbb{Y}^2} \sigma^2(r)\sigma^2(s) N^{1-2d(r,s)} B(d(s), 1-2d(s))\mu(dr)\mu(ds) \qquad (5.32)\\
\leqslant \int_{\mathbb{Y}} \sigma^2(r)N^{1/2-2d(r)}\mu(dr)\int_{\mathbb{Y}} \sigma^2(s)N^{1/2-2d(s)}\frac{1}{d(s)\,(1-2d(s))}\mu(ds)\\
\leqslant 4\int_{\mathbb{Y}} \sigma^2(r)N^{1/2-2d(r)}\mu(dr)\int_{\mathbb{Y}} N^{1/2-2d(s)}\frac{\sigma^2(s)}{1-2d(s)}\mu(ds),
\end{aligned}
$$

where the third inequality follows from the bound $0 \leqslant B(x,y) \leqslant \frac{1}{xy}$; see Theorem 1.1 in From and Ratnasingam (2022). Then, $\mathcal{R}_{2,N} \to 0$ as $N \to \infty$ from Assumption (3.13).

The calculations above imply that

$$
\begin{aligned}
\left\| \widetilde{C}_N^{\sigma} - \bar{\mathfrak{f}}^{\sigma}\right\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))} &\leqslant 2\left\|\widetilde{C}_N^{\sigma}\right\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))} + 2\left\|\bar{\mathfrak{f}}^{\sigma}\right\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))}\\
&\leqslant 10\left\|\bar{\mathfrak{f}}^{\sigma}\right\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))} + 8\sum_{i=1}^{3}\mathcal{R}_{i,N} < \infty. \qquad (5.33)
\end{aligned}
$$

We turn to point (ii). Note that from the definition of $\{u_j\}_{j\in\mathbb{N}}$, for all $v, x_1, x_2 \in \mathbb{R}$ and $r, s \in \mathbb{Y}$, as $N \to \infty$

$$
\begin{aligned}
N^{1-d(r)}u_{\lceil vN\rceil + h - \lceil x_1 N\rceil}(r)\mathbb{1}_{\{n+h-j_1\geqslant 0\}}\sigma(r) &\to (v-x_1)_+^{d(r)-1}\sigma(r),\\
N^{1-d(s)}u_{\lceil vN\rceil - \lceil x_2 N\rceil}(s)\mathbb{1}_{\{n-j_2\geqslant 0\}}\sigma(s) &\to (v-x_2)_+^{d(s)-1}\sigma(s).
\end{aligned} \qquad (5.34)
$$

The a.s. pointwise convergence (in (r, s, x_1, x_2) of $\widetilde{C}_{N,h}(x_1, x_2, r, s)$ to $\mathfrak{f}^{\sigma}$) follows by DCT by using similar bounds as in the proof of (i), and noticing that these bounds are finite for $\lambda^{\otimes 2}\otimes\mu^{\otimes 2}$ a.s. (x_1, x_2, r, s).

Combining items (i), (ii), and DCT, we can infer that

$$
\|\widetilde{C}_N^{\sigma} - \bar{\mathfrak{f}}^{\sigma}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{H+1}))} \to 0 \qquad (5.35)
$$

This shows that the conditions in Lemma 6.3 are satisfied, finishing the proof of (5.6). □

5.3 Proof of Theorem 3.5

The proof of Theorem 3.5 is based on Theorem 3.3. We show that the $\mathbb{H}$-valued linear process (1.1) with (1.5) can be written as a continuous operator applied to an $L^2(\mathbb{Y})$-valued linear process (1.1). Analogously, the sample autocovariance operator can be written as a continuous operator applied to the sample autocovariances of an $L^2(\mathbb{Y})$-valued linear process. Then, the continuous mapping theorem and Theorem 3.3 give the desired result.

Recall the spectral theorem from (2.6). The self-adjoint operator T can be decomposed into a multiplication operator D_d and a unitary operator U. Then,

$$\begin{aligned} X_n = \sum_{j=0}^{\infty}(j+1)^{T-I}\varepsilon_{n-j} &= \sum_{j=0}^{\infty}\exp\left(U^*(D_d-I)U\log(j+1)\right)\varepsilon_{n-j} \\ &= \sum_{j=0}^{\infty}\sum_{\kappa=0}^{\infty}U^*\frac{(D_d-I)^\kappa\log(j+1)^\kappa}{\kappa!}U\varepsilon_{n-j} \\ &= U^*\left(\sum_{j=0}^{\infty}(j+1)^{D_d-I}(U\varepsilon_{n-j})\right), \end{aligned} \tag{5.36}$$

where we used, for $T\in L(\mathbb{H})$ and $\lambda>0$, $\lambda^T=e^{T\log(\lambda)}$ and $e^T=\sum_{j=0}^{\infty}T^j/j!$ in the first and second line, respectively. Therefore,

$$X_n = U^*Z_n \quad \text{with} \quad Z_n \doteq \sum_{j=0}^{\infty}(j+1)^{D_d-I}(U\varepsilon_{n-j}) \in L^2(\mathbb{Y}). \tag{5.37}$$

Recall from Section 2.3 that the decomposition (5.37), allows us to infer some properties of $\{X_n\}_{n\in\mathbb{Z}}$ based on Z_n. In particular, if Z_n satisfies (2.10) with $\sigma^2(r)=\mathrm{E}((U\varepsilon(r))(U\varepsilon(r)))$, the unitarity of U implies that $\{X_n\}_{n\in\mathbb{Z}}$ converges P-almost surely.

The interchangeability of the series and U^* in (5.36) is a consequence of the almost sure convergence of Z_n in (5.37) and the boundedness of the unitary operator U. The process $\{Z_n\}_{n\in\mathbb{Z}}$ satisfies the assumptions of Theorem 3.3: The sequence $\{U\varepsilon_j\}_{j\in\mathbb{Z}}$ is an i.i.d. sequence with finite second and fourth moments since $U:\mathbb{H}\to L^2(\mathbb{Y})$ is a unitary operator and $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ is assumed to be an i.i.d. sequence with finite second and fourth moments.

In view of (5.37), we can rewrite the sample autocovariance operators of lag h as

$$\begin{aligned} \widehat{\Gamma}_{N,h}-\Gamma_h &= \frac{1}{N}\sum_{n=1}^{N}X_{n+h}\otimes X_n - \mathrm{E}(X_h\otimes X_0) \\ &= \frac{1}{N}\sum_{n=1}^{N}((U^*Z_{n+h})\otimes(U^*Z_n) - E((U^*Z_h)\otimes(U^*Z_0))) \\ &= \frac{1}{N}\sum_{n=1}^{N}((U^*\otimes U^*)(Z_{n+h}\otimes Z_n) - (U^*\otimes U^*)\,\mathrm{E}(Z_h\otimes Z_0)) \\ &= (U^*\otimes U^*)\left[\frac{1}{N}\sum_{n=1}^{N}(Z_{n+h}\otimes Z_n - \mathrm{E}(Z_h\otimes Z_0))\right]. \end{aligned}$$

To interchange the summation and integration operations with U in the calculations above, we used the fact that U is a unitary, bounded, and linear operator.

Recall the scaling operators Ξ_N, Δ_N^U defined in (3.11) and (3.14). It follows that the normalized sample autocovariance operator (1.2) of X_n can be written as

$$\begin{aligned} \Delta_N^U(\widehat{\Gamma}_{N,h}-\Gamma_h) &= (U^*\otimes U^*)\left[\Xi_N\left(\frac{1}{N}\sum_{n=1}^{N}(Z_{n+h}\otimes Z_n - \mathrm{E}(Z_h\otimes Z_0))\right)\right] \\ &\xrightarrow{d} (U^*\otimes U^*)\mathcal{I}_2^{\sigma_U}(f) \end{aligned} \tag{5.38}$$

with

$$\sigma_U(r,s) \doteq \mathrm{E}\left(((U\varepsilon)(r))((U\varepsilon)(s))\right), \quad r,s\in\mathbb{Y}.$$

The weak convergence

$$\Xi_N \left(\frac{1}{N} \sum_{n=1}^{N} (Z_{n+h} \otimes Z_n - \mathrm{E}(Z_h \otimes Z_0)) \right) \xrightarrow{d} \mathcal{I}_2^{\sigma_U}(f)$$

is in $L^2(\mathbb{Y}^2)$ and follows from an application of Theorem 3.3. The weak convergence in (5.38) is in $\mathbb{H} \otimes \mathbb{H}$ and holds from the continuous mapping theorem, since $U^* \otimes U^*$ is bounded and linear. Note that the second line in (5.38) includes a slight abuse in notation, by identifying $L^2(\mathbb{Y}^2) \cong L^2(\mathbb{Y}) \otimes L^2(\mathbb{Y})$.

We can conclude that the continuous mapping theorem, (5.38) and Lemma 3.3 give the desired convergence result.

6 Some Technical Results

In this section, we give some technical results and their proofs. We start with a lemma providing a technical estimate.

Lemma 6.1. *The function $c(r,s)$ in (2.2) can be bounded from above as*

$$c(r,s) \leqslant \frac{1}{d(r)} + \frac{1}{1-d(r,s)} \tag{6.1}$$

for $d(r,s) = d(r) + d(s)$ and $d(s) \in (0, \frac{1}{2})$.

Proof: As a function of d, the Beta function can be written as

$$c(r,s) = \int_0^\infty x^{d(r)-1}(x+1)^{d(s)-1} dx = \int_0^1 x^{d(r)-1}(1-x)^{-d(r,s)} dx. \tag{6.2}$$

Note that $ab \leqslant a + b$ for $a, b > 0$, which implies

$$x^{d(r)-1}(1-x)^{-d(r,s)} \leqslant x^{d(r)-1} + (1-x)^{-d(r,s)}. \tag{6.3}$$

Then, (6.3) yields

$$c(r,s) \leqslant \int_0^1 x^{d(r)-1} dx + \int_0^1 (1-x)^{-d(r,s)} dx \leqslant \frac{1}{d(r)} + \frac{1}{1-d(r,s)}, \tag{6.4}$$

concluding the proof of the Lemma. □

The following lemma ensures that, in the second regime, the conditions of Theorem 3.5 imply the desired regularity for the kernels f in (4.14).

Lemma 6.2. *Recall the kernel $\mathfrak{f}$ from (4.14) and let $d(r) \in \left(\frac{1}{4}, \frac{1}{2}\right)$. If*

$$\int_{\mathbb{Y}} \frac{\sigma^2(r)}{(1-2d(r))(2d(r)-1/2)} \mu(dr) < \infty, \tag{6.5}$$

then $\mathfrak{f}^\sigma(r,s,x,y) = \mathfrak{f}^{(r,s)}(x,y)\sigma(r)\sigma(s)$ takes values in $L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2))$.

Proof: First, note that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left|\int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\right|^2 dx_1dx_2\\
&=\int_{\mathbb{R}^2}\left(\int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\right)\left(\int_0^1 (u-x_1)_+^{d(r)-1}(u-x_2)_+^{d(s)-1}du\right)dx_1dx_2\\
&=\int_0^1\int_0^1\left(\int_{\mathbb{R}} (v-x_1)_+^{d(r)-1}(u-x_1)_+^{d(r)-1}dx_1\right)\left(\int_{\mathbb{R}} (v-x_2)_+^{d(s)-1}(u-x_2)_+^{d(s)-1}dx_2\right)dvdu\\
&=\int_0^1\int_0^1\left(\int_{\mathbb{R}} (z_1)_+^{d(r)-1}(u-v+z_1)_+^{d(r)-1}dz_1\right)\left(\int_{\mathbb{R}} (z_2)_+^{d(s)-1}(u-v+z_2)_+^{d(s)-1}dz_2\right)dvdu, \quad (6.6)
\end{aligned}
$$

where (6.6) follows by substituting $x_1 = v - z_1$ and $x_2 = v - z_2$. We want to perform the change of variables $z_i = (u-v)w_i, i = 1, 2$. For this, we first see that, for the sets

$$
\begin{aligned}
&\mathcal{A} \doteq \{z_1 \geqslant 0, z_2 \geqslant 0\} = \{u \geqslant v, w_1 \geqslant 0, w_2 \geqslant 0\} \cup \{u \leqslant v, w_1 \leqslant 0, w_2 \leqslant 0\}\\
&\mathcal{B} \doteq \{u-v+z_1 \geqslant 0, u-v+z_2 \geqslant 0\} = \{u \geqslant v, w_1 \geqslant -1, w_2 \geqslant -1\} \cup \{u \leqslant v, w_1 \leqslant -1, w_2 \leqslant -1\},
\end{aligned}
$$

we have that

$$
\mathcal{A}\cap\mathcal{B} = \{u \geqslant v, w_1 \geqslant 0, w_2 \geqslant 0\} \cup \{u \leqslant v, w_1 \leqslant -1, w_2 \leqslant -1\}. \quad (6.7)
$$

In view of (6.7) and the change of variables $z_i = (u-v)w_i, i = 1, 2$, (6.6) implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left|\int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\right|^2 dx_1dx_2\\
&=\int_0^1\int_0^u (u-v)^{2d(r,s)-2}dvdu\\
&\qquad\times\int_0^\infty (w_1)^{d(r)-1}(1+w_1)^{d(r)-1}dw_1\int_0^\infty (w_2)^{d(s)-1}(1+w_2)^{d(s)-1}dw_2 \quad (6.8)\\
&\quad+\int_0^1\int_u^1 (v-u)^{2d(r,s)-2}dvdu\\
&\qquad\times\int_{-\infty}^{-1} (-w_1)^{d(r)-1}(-1-w_1)^{d(r)-1}dw_1\int_{-\infty}^{-1} (-w_2)^{d(s)-1}(-1-w_2)^{d(s)-1}dw_2 \quad (6.9)\\
&=2\int_0^\infty (w_1)^{d(r)-1}(1+w_1)^{d(r)-1}dw_1\int_0^\infty (w_2)^{d(s)-1}(1+w_2)^{d(s)-1}dw_2\\
&\qquad\times\left(\int_0^1\left(\int_0^u (u-v)^{2d(r,s)-2}+\int_u^1 (v-u)^{2d(r,s)-2}\right)dvdu\right), \quad (6.10)
\end{aligned}
$$

where (6.10) follows by the change of variables $x_i = -1 - w_i, i = 1, 2$ of the integrals in (6.9) (we still denote the new variable by w_i). By recalling the definition of $c(r) \doteq c(r,r)$, in (2.2) and elementary calculations, this says that

$$
\begin{aligned}
\int_{\mathbb{R}^2}\left|\int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\right|^2 dx_1dx_2 &= 4c(r)c(s)\frac{1}{2d(r,s)-1}\frac{1}{2d(r,s)}\\
&\leqslant 4c(r)c(s)\frac{1}{2d(r,s)-1},
\end{aligned} \quad (6.11)
$$

since $d(r) > 1/4$.

We see that, by Tonelli's theorem,

$$\|\mathfrak{f}^\sigma\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2))} = \int_{\mathbb{R}^2}\left(\int_{\mathbb{Y}^2}\left|\sigma(r)\sigma(s)\int_0^1 (v-x_1)_+^{d(r)-1}(v-x_2)_+^{d(s)-1}dv\right|^2 \mu(dr)\mu(ds)\right)dx_1dx_2$$

$$\leqslant 4\int_{\mathbb{Y}^2}\sigma^2(r)\sigma^2(s)c(r)c(s)\frac{1}{2d(r,s)-1}\mu(dr)\mu(ds) \tag{6.12}$$

$$\leqslant 4\int_{\mathbb{Y}^2}\left(\frac{1}{d(r)}+\frac{1}{1-2d(r)}\right)\left(\frac{1}{d(s)}+\frac{1}{1-2d(s)}\right)\frac{\sigma^2(r)\sigma^2(s)}{2d(r,s)-1}\mu(dr)\mu(ds) \tag{6.13}$$

$$\leqslant 36\left(\int_{\mathbb{Y}}\frac{\sigma^2(r)}{(1-2d(r))(2d(r)-1/2)}\mu(dr)\right)^2, \tag{6.14}$$

where (6.12) is due to (6.11), (6.13) follows from (6.1), and (6.14) follows by recalling that, for $1/4 \leqslant d(r), d(s) \leqslant 1/2$, we have the relations

$$\frac{1}{2d(r,s)-1} = \frac{1}{2d(r)-1/2+2d(s)-1/2} \leqslant \frac{1}{2d(r)-1/2}\frac{1}{2d(s)-1/2}, \quad \frac{1}{d(r)} \leqslant \frac{2}{1-2d(r)}.$$

The proof is concluded upon noticing that the quantity in (6.14) is finite. □

The following key lemma provides the necessary ingredients to obtain convergence to double Wiener-Itô integrals with sample paths in $L^2(\mathbb{Y}^2)$. It generalizes Proposition 14.3.2 in Giraitis et al. (2012).

Lemma 6.3. *Let C_N be as in (5.19). Consider a linear combination of an off-diagonal tuple*

$$Q_2(C_N) \in L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)}), \quad Q_2(C_N)(r,s) \doteq \sum_{j_1\neq j_2} C_N(j_1,j_2,r,s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s).$$

Assume that there exists a kernel $f \in \mathcal{H}^2$, such that, denoting $f^\sigma(r,s,x,y) \doteq f(r,s,x,y)\sigma(r)\sigma(s)$ and $\bar{f} \doteq (f,\ldots,f) \in L^2(\mathbb{R}^2 : L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)}))$, the functions defined in (5.20) satisfy, as $N \to \infty$,

$$\|\widetilde{C}^\sigma_N - \bar{f}^\sigma\|_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \to 0. \tag{6.15}$$

Then,

$$Q_2(C_N) \xrightarrow{\mathrm{d}} \mathcal{I}_2(\bar{f}) \doteq \begin{pmatrix}\mathcal{I}_2^\sigma(f)\\ \vdots \\ \mathcal{I}_2^\sigma(f)\end{pmatrix}, \quad \textit{weakly in } L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)}),$$

where each coordinate of the limit is understood in the sense of Definition 4.2.

Proof: For notational simplicity, we write from here on $\mathcal{I}_2(f)$ instead of $\mathcal{I}_2^\sigma(f)$. By a truncation argument, it is enough to prove that for all $\varepsilon > 0$, there exists a special kernel f_ε (cf. Section 4), its vectorization $\bar{f}_\varepsilon$, and a corresponding $C_{N,\varepsilon}$, such that

$$\operatorname{Var}\|Q_2(C_N) - Q_2(C_{N,\varepsilon})\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \leqslant \varepsilon, \tag{6.16}$$

$$\operatorname{Var}\|\mathcal{I}_2(\bar{f}_\varepsilon) - \mathcal{I}_2(\bar{f})\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \leqslant \varepsilon, \tag{6.17}$$

$$Q_2(C_{N,\varepsilon}) \xrightarrow{d} \mathcal{I}_2(\bar{f}_\varepsilon), \tag{6.18}$$

as $N \to \infty$, where the weak convergence in (6.18) is in the topology of $L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$ and $C_{N,\varepsilon}$ is defined by

$$C_{N,\varepsilon}(j_1,j_2,r,s) \doteq N^{-1}\bar{f}_\varepsilon^{(r,s)}\left(\frac{j_1}{N},\frac{j_2}{N}\right), \quad r,s \in \mathbb{Y}. \tag{6.19}$$

Indeed, if (6.16)–(6.18) are true, then we can write, for each $\varepsilon > 0$

$$\begin{aligned} Q_2(C_N) - \mathcal{I}_2(\bar{f}) &= Q_2(C_N) - Q_2(C_{N,\varepsilon}) + Q_2(C_{N,\varepsilon}) - \mathcal{I}_2(\bar{f}_\varepsilon) + \mathcal{I}_2(\bar{f}_\varepsilon) - \mathcal{I}_2(\bar{f}) \\ &= \mathcal{A}_{N,\varepsilon} + \mathcal{B}_{N,\varepsilon} + \mathcal{C}_{N,\varepsilon}, \end{aligned} \tag{6.20}$$

where

$$\mathcal{A}_{N,\varepsilon} \doteq Q_2(C_N) - Q_2(C_{N,\varepsilon}), \quad \mathcal{B}_{N,\varepsilon} \doteq Q_2(C_{N,\varepsilon}) - \mathcal{I}_2(\bar{f}_\varepsilon), \quad \mathcal{C}_{N,\varepsilon} \doteq \mathcal{I}_2(\bar{f}_\varepsilon) - \mathcal{I}_2(\bar{f}). \tag{6.21}$$

Then, by Condition (6.16) it follows that $\mathcal{A}_{N,\varepsilon} \xrightarrow{P} 0$ in $L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$. Moreover, by Condition (6.18), it follows that $\mathcal{B}_{\mathcal{N},\varepsilon} \to 0$ in distribution, and so in probability. Finally, (6.17) implies that $\mathcal{C}_{\mathcal{N},\varepsilon} \xrightarrow{P} 0$. We can then conclude that

$$Q_2(C_N) \to \mathcal{I}_2(\bar{f}), \quad \text{weakly in } L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)}), \tag{6.22}$$

from Slutsky's lemma, by sending first $N \to \infty$, and then $\varepsilon \to 0$.

We turn to verifying conditions (6.16)–(6.18). First note that, whenever $i_1 \neq i_2, j_1 \neq j_2$,

$$\mathrm{E}(\varepsilon_{i_1}(r_1)\varepsilon_{j_1}(s_1)\varepsilon_{i_2}(r_2)\varepsilon_{j_2}(s_2)) = \begin{cases} \sigma(r_1, s_1)\sigma(r_2, s_2), & \text{if } i_1 = j_1, i_2 = j_2, \\ \sigma(r_1, s_2)\sigma(s_1, r_2), & \text{if } i_1 = j_2, j_1 = i_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for (6.16),

$$\begin{aligned} &\mathrm{E}\,\|Q_2(C_N)\|^2_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \\ &= \mathrm{E}\Big(\int_{\mathbb{Y}}\int_{\mathbb{Y}}\Big\|\sum_{j_1\neq j_2} C_N(j_1, j_2, r, s)\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\Big\|^2_{\mathbb{R}^{H+1}}\mu(dr)\mu(ds)\Big) \\ &= \int_{\mathbb{Y}}\int_{\mathbb{Y}}\sum_{j_1\neq j_2}\sum_{i_1\neq i_2}(C_N(j_1, j_2, r, s))'C_N(i_1, i_2, r, s)\,\mathrm{E}(\varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\varepsilon_{i_1}(r)\varepsilon_{i_2}(s))\mu(dr)\mu(ds) \\ &= \int_{\mathbb{Y}}\int_{\mathbb{Y}}\sum_{j_1\neq j_2}\|C_N(j_1, j_2, r, s)\|^2\sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds) \\ &\qquad + \int_{\mathbb{Y}}\int_{\mathbb{Y}}\sum_{j_1\neq j_2}|C_N(j_1, j_2, r, s)|'|C_N(j_2, j_1, r, s)|\sigma^2(r, s)\mu(dr)\mu(ds) \\ &\leqslant 2\int_{\mathbb{Y}}\int_{\mathbb{Y}}\int_{\mathbb{R}^2} N^2\|C_N(\lceil x_1 N\rceil, \lceil x_2 N\rceil, r, s)\|^2 dx_1 dx_2 \sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds) \qquad (6.23) \\ &\leqslant 2\|\widetilde{C}^\sigma_N\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \qquad (6.24) \end{aligned}$$

with $\widetilde{C}^\sigma_N(x_1, x_2, r, s)$ as in (5.20), and where (6.23) holds by noticing that $\sigma^2(r, s) \leqslant \sigma^2(r)\sigma^2(s)$. This implies

$$\begin{aligned} \mathrm{E}\,\|Q_2(C_N) - Q_2(C_{N,\varepsilon})\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} &= \mathrm{E}\,\|Q_2(C_N - C_{N,\varepsilon})\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \\ &\leqslant 2\|\widetilde{C}^\sigma_N - \widetilde{C}^\sigma_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}, \end{aligned} \tag{6.25}$$

where

$$\widetilde{C}^\sigma_{N,\varepsilon}(x_1, x_2, r, s) \doteq NC_{N,\varepsilon}(x_1, x_2, r, s)\sigma(r)\sigma(s). \tag{6.26}$$

It remains to bound the right hand side of (6.25). First note that if $f \in \mathcal{H}^2$ is special, then so is f^σ. Then, for any $N_0 \geqslant 1$ and any special $\bar{f}^\sigma_\varepsilon, \widetilde{C}^\sigma_{N,\varepsilon}$, by the triangle and Cauchy-Schwarz inequalities,

$$\|\widetilde{C}^\sigma_N - \widetilde{C}^\sigma_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \leqslant 3\|\widetilde{C}^\sigma_N - \widetilde{C}^\sigma_{N_0}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}$$

$$+ 3\|\widetilde{C}^{\sigma}_{N_0} - \bar{f}^{\sigma}_{\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} + 3\|\bar{f}^{\sigma}_{\varepsilon} - \widetilde{C}^{\sigma}_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}. \tag{6.27}$$

By assumption, there is a $N_0 \geqslant 1$ such that, for the choice of kernels f^{σ}

$$\begin{aligned}&\|\widetilde{C}^{\sigma}_{N} - \widetilde{C}^{\sigma}_{N_0}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}\\ &\leqslant 2\|\widetilde{C}^{\sigma}_{N} - \bar{f}^{\sigma}\|^2_{L^2(\mathbb{R}^2:(L^2(\mathbb{Y}^2)^{\times(H+1)}))} + 2\|\bar{f}^{\sigma} - \widetilde{C}^{\sigma}_{N_0}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \leqslant \frac{\varepsilon}{18}\end{aligned} \tag{6.28}$$

for all $N \geqslant N_0$. Given $N_0 \geqslant 1$ and $\varepsilon > 0$, since $f \in \mathcal{H}^2$, there exist special kernels $f_{\varepsilon} \in \mathcal{H}^2$ as in (4.6) (with c replaced by $\bar{c}$) and their induced vectorization $\bar{f}^{\sigma}_{\varepsilon}$ from Theorem 4.4 such that

$$\|\widetilde{C}^{\sigma}_{N_0} - \bar{f}^{\sigma}_{\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \leqslant \|\widetilde{C}^{\sigma}_{N_0} - \bar{f}^{\sigma}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} + \|\bar{f}^{\sigma} - \bar{f}^{\sigma}_{\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \leqslant \varepsilon/9. \tag{6.29}$$

Moreover, the function $\widetilde{C}_{N,\varepsilon}$ derived from $C_{N,\varepsilon}$ satisfies, for all $\varepsilon > 0$,

$$\begin{aligned}&\|\bar{f}^{\sigma}_{\varepsilon} - \widetilde{C}^{\sigma}_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}\\ &= \int_{\mathbb{R}^2} \left\| \bar{f}^{\sigma}_{\varepsilon}(x_1, x_2) - \bar{f}^{\sigma}_{\varepsilon}\Big(\frac{\lceil x_1 N\rceil}{N}, \frac{\lceil x_2 N\rceil}{N}\Big) \right\|^2_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} dx_1 dx_2\\ &\leqslant (H+1) \int_{\mathbb{Y}^2} \sigma^2(r)\sigma^2(s)\|\bar{c}_{i_1,i_2}(r,s)\|^2 \mu(dr)\mu(ds)\\ &\qquad \times \int_{\mathbb{R}^2} \left(\mathbb{1}_{\Delta_{i_1}}(x_1)\mathbb{1}_{\Delta_{i_2}}(x_2) - \mathbb{1}_{\Delta_{i_1}}\Big(\frac{\lceil x_1 N\rceil}{N}\Big)\mathbb{1}_{\Delta_{i_2}}\Big(\frac{\lceil x_2 N\rceil}{N}\Big)\right) dx_1 dx_2 \to 0,\end{aligned}$$

as $N \to \infty$, where we used that f^{σ}_{ε} is a special kernel and DCT. Therefore, we can choose an $N_1 \geqslant N_0$ such that, for all $N \geqslant N_1$,

$$\|\bar{f}^{\sigma}_{\varepsilon} - \widetilde{C}^{\sigma}_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \leqslant \varepsilon/9. \tag{6.30}$$

Combining the relations in (6.27), (6.28), (6.29), and (6.30) finishes the proof of (6.16).

We move to Condition (6.17). Note that

$$\begin{aligned}&\operatorname{Var}\|\mathcal{I}_2(\bar{f}_{\varepsilon}) - \mathcal{I}_2(\bar{f})\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})}\\ &= \mathrm{E}\int_{\mathbb{Y}^2}\int_{\mathbb{R}^4} (\bar{f}^{(r,s)}(x_1,x_2) - \bar{f}^{(r,s)}_{\varepsilon}(x_1,x_2))'(\bar{f}^{(r,s)}(y_1,y_2) - \bar{f}^{(r,s)}_{\varepsilon}(y_1,y_2))\\ &\qquad\qquad \times W^{(r)}(dx_1)W^{(s)}(dx_2)W^{(r)}(dy_1)W^{(s)}(dy_2)\mu(dr)\mu(ds)\\ &= \int_{\mathbb{Y}^2}\int_{\mathbb{R}^2} \|\bar{f}^{(r,s)}(x_1,x_2) - \bar{f}^{(r,s)}_{\varepsilon}(x_1,x_2)\|^2 dx_1 dx_2 \sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds)\\ &\quad + \int_{\mathbb{Y}^2}\int_{\mathbb{R}^2} (\bar{f}^{(r,s)}(x_1,x_2) - \bar{f}^{(r,s)}_{\varepsilon}(x_1,x_2))'(\bar{f}^{(r,s)}(x_2,x_1) - \bar{f}^{(r,s)}_{\varepsilon}(x_2,x_1)) dx_1 dx_2\\ &\qquad \times \sigma^2(r,s)\mu(dr)\mu(ds)\\ &\leqslant 2\int_{\mathbb{Y}^2}\int_{\mathbb{R}^2} \|\bar{f}^{(r,s)}(x_1,x_2) - \bar{f}^{(r,s)}_{\varepsilon}(x_1,x_2)\|^2 dx_1 dx_2 \sigma^2(r)\sigma^2(s)\mu(dr)\mu(ds),\end{aligned} \tag{6.31}$$

where the last inequality follows by Young's inequality. Moreover, by Theorem 4.4 we can choose $f^{(r,s)}_{\varepsilon}(x,y)$ to be dominated from $f^{(r,s)}(x,y)$ pointwise. This says that, from Lemma 6.2 and Cauchy-Schwarz,

$$\operatorname{Var}\|\mathcal{I}_2(\bar{f}_{\varepsilon}) - \mathcal{I}_2(\bar{f})\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \leqslant 3\|\bar{f}^{\sigma} - \bar{f}^{\sigma}_{\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}.$$

Then, by DCT, we can select f_{ε} such that, in addition to (6.16),

$$\operatorname{Var}\|\mathcal{I}_2(\bar{f}_{\varepsilon}) - \mathcal{I}_2(\bar{f})\|_{L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)})} \leqslant \varepsilon. \tag{6.32}$$

For (6.18), note that

$$Q_2(C_{N,\varepsilon}) = \sum_{j_1 \neq j_2} C_{N,\varepsilon}(j_1, j_2, \cdot, \cdot)\varepsilon_{j_1}(\cdot)\varepsilon_{j_2}(\cdot) = \sum_{j_1 \neq j_2} N^{-1} \bar{f}_\varepsilon^{(\cdot,\cdot)}\Big(\frac{j_1}{N}, \frac{j_2}{N}\Big)\varepsilon_{j_1}(\cdot)\varepsilon_{j_2}(\cdot).$$

In order to show that $Q_2(C_{N,\varepsilon})$ converges weakly to $\mathcal{I}_2(\bar{f}_\varepsilon)$ in $L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$, we employ Theorem 2 in Cremers and Kadelka (1986). We rephrase the conditions for weak convergence here. A sequence ξ_n converges weakly to $\xi \in L^2(\mathbb{Y}^2 : \mathbb{R}^{(H+1)})$ if

(i) the finite-dimensional distributions converge, i.e.,

$$(\xi_n(r_1, s_1), \ldots, \xi_n(r_q, s_q))' \xrightarrow{d} (\xi(r_1, s_1), \ldots, \xi(r_q, s_q))', \tag{6.33}$$

in $\mathbb{R}^{q\times(H+1)}$ for all $r_1, \ldots, r_q, s_1, \ldots, s_q \in \mathbb{Y}$ and $q \in \mathbb{N}$.

(ii) For all $r, s \in \mathbb{Y}$ we have that $\mathrm{E}\,\|\xi_n(r,s)\|^2_{\mathbb{R}^{(H+1)}} \to \mathrm{E}\,\|\xi(r,s)\|^2_{\mathbb{R}^{(H+1)}}$.

(iii) there exists a $\mu \otimes \mu$-integrable function $g : \mathbb{Y}^2 \to \mathbb{R}$ such that $\mathrm{E}\,\|\xi_n(r,s)\|^2_{\mathbb{R}^{H+1}} \leqslant g(r,s)$ for each $(r,s) \in \mathbb{Y}^2$.

Proof of condition (i): Since $\bar{f}_\varepsilon$ are special, they take non-zero values in a finite number of intervals $\{\Delta_i\}_{i=1,\ldots,J}$, and so can be rewritten as

$$\bar{f}_\varepsilon^{(r,s)}(x,y) = \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{J} \bar{c}_{i_1,i_2}(r,s)\mathbb{1}_{\Delta_{i_1}}(x)\mathbb{1}_{\Delta_{i_2}}(y), \tag{6.34}$$

with suitable functions $c_{i_1,i_2} : \mathbb{Y}^2 \to \mathbb{R}$, $\bar{c} = (c, \ldots, c) \in L^2(\mathbb{Y}^2, \mathbb{R}^{H+1})$. Then, for $r, s \in \mathbb{Y}$, from (6.19),

$$\begin{aligned} Q_2(C_{N,\varepsilon})(r,s) &= \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{J} \bar{c}_{i_1,i_2}(r,s)N^{-1}\sum_{j_1 \neq j_2} \varepsilon_{j_1}(r)\varepsilon_{j_2}(s)\mathbb{1}_{\{\frac{j_1}{N}\in\Delta_1, \frac{j_2}{N}\in\Delta_2\}} \\ &= \sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{J} \bar{c}_{i_1,i_2}(r,s)W_N^{(r)}(\Delta_{i_1})W_N^{(s)}(\Delta_{i_2}), \end{aligned} \tag{6.35}$$

where

$$W_N^{(r)}(\Delta_i) \doteq N^{-\frac{1}{2}} \sum_{j:\frac{j}{N}\in\Delta_i} \varepsilon_j(r) = N^{-\frac{1}{2}} \sum_{j=1}^{N} \varepsilon_j(r)\mathbb{1}_{\{\frac{j}{N}\in\Delta_i\}}.$$

Similarly,

$$\begin{pmatrix} Q_2(C_{N,\varepsilon})(r_1,s_1) \\ \vdots \\ Q_2(C_{N,\varepsilon})(r_q,s_q) \end{pmatrix} = \begin{pmatrix} \sum_{i_1\neq i_2}^{J} \bar{c}_{i_1,i_2}(r_1,s_1)W_N^{(r_1)}(\Delta_{i_1})W_N^{(s_1)}(\Delta_{i_2}) \\ \vdots \\ \sum_{i_1\neq i_2}^{J} \bar{c}_{i_1,i_2}(r_q,s_q)W_N^{(r_q)}(\Delta_{i_1})W_N^{(s_q)}(\Delta_{i_2}) \end{pmatrix}. \tag{6.36}$$

We study the joint weak convergence of $W_N^{(x)}(\Delta_p)$, for $p = 1, \ldots, J$ and $x = r_1, \ldots, r_q, s_1, \ldots, s_q$. Denote

$$W_N \doteq \begin{pmatrix} W_N(\Delta_1) \\ \vdots \\ W_N(\Delta_J) \end{pmatrix}, \quad W_N(\Delta_p) \doteq \begin{pmatrix} W_N^{(r_1)}(\Delta_p) \\ W_N^{(s_1)}(\Delta_p) \\ \vdots \\ W_N^{(r_q)}(\Delta_p) \\ W_N^{(s_q)}(\Delta_p) \end{pmatrix}, \quad p = 1, \ldots J. \tag{6.37}$$

Note that from a standard multivariate CLT, we have, for $p = 1, \ldots, J$,

$$W_N(\Delta_p) \xrightarrow{d} W(\Delta_p) \in \mathbb{R}^{2q}, \quad \Sigma_p \doteq \begin{pmatrix} \sigma^2(r_1) & \sigma(r_1, s_1) & \sigma(r_1, r_2) & \ldots & \sigma(r_1, s_q) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma(s_q, r_1) & \sigma(s_q, s_1) & \sigma(s_q, r_2) & \ldots & \sigma^2(s_q) \end{pmatrix},$$

where $W(\Delta_p)$ is a Gaussian random vector with covariance matrix Σ_p. Moreover, $W(\Delta_p)$ and $W(\Delta_s)$ are independent whenever $p \neq s$ since Δ_p and Δ_s are disjoint. We therefore obtain the joint convergence, as $N \to \infty$,

$$W_N \xrightarrow{d} W \tag{6.38}$$

where W is a $\mathbb{R}^{J\times(2q)}$-valued Gaussian random variable that has a covariance matrix given by the block diagonal form $\Sigma_{m,n} = \Sigma_m \delta_{m,n}, m, n = 1, \ldots J$.

Then, we can recast

$$\begin{pmatrix} Q_2(C_{N,\varepsilon})(r_1, s_1) \\ \vdots \\ Q_2(C_{N,\varepsilon})(r_q, s_q) \end{pmatrix} = D(W_N), \tag{6.39}$$

where W_N is as in (6.37) and D is a suitable function involving $c_{i_1,i_2}, i_1, i_2 = 1, \ldots J$ (see, Giraitis et al. (2012) pp. 535-536). We then obtain by the continuous mapping theorem that, as $N \to \infty$,

$$\begin{pmatrix} Q_2(C_{N,\varepsilon})(r_1, s_1) \\ \vdots \\ Q_2(C_{N,\varepsilon})(r_q, s_q) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \sum_{i_1\neq i_2} \bar{c}_{i_1,i_2}(r_1, s_1) W^{(r_1)}(\Delta_{i_1}) W^{(s_1)}(\Delta_{i_2}) \\ \vdots \\ \sum_{i_1\neq i_2} \bar{c}_{i_1,i_2}(r_q, s_q) W^{(r_q)}(\Delta_{i_1}) W^{(s_q)}(\Delta_{i_2}) \end{pmatrix} = \begin{pmatrix} \mathcal{I}_2(\bar{f}_\varepsilon)(r_1, s_1) \\ \vdots \\ \mathcal{I}_2(\bar{f}_\varepsilon)(r_q, s_q) \end{pmatrix} \in \mathbb{R}^{q\times(H+1)}.$$

Proof of condition (ii): For fixed $r, s \in \mathbb{Y}$, we have

$$\begin{aligned}
&\mathrm{E}\, \|\mathcal{I}_2(\bar{f}_\varepsilon)(r, s)\|^2_{\mathbb{R}^{(H+1)}} \\
&= \mathrm{E}\left(\int_{\mathbb{R}^4} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(y_1, y_2) W^{(r)}(dx_1) W^{(s)}(dx_2) W^{(r)}(dy_1) W^{(s)}(dy_2) \right) \\
&= \int_{\mathbb{R}^4} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(y_1, y_2)\, \mathrm{E}\left(W^{(r)}(dx_1) W^{(s)}(dx_2) W^{(r)}(dy_1) W^{(s)}(dy_2) \right) \\
&= \int_{\mathbb{R}^2} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(x_1, x_2)\, \mathrm{E}\left(W^{(r)}(dx_1) W^{(r)}(dx_1) \right) \mathrm{E}\left(W^{(s)}(dx_2) W^{(s)}(dx_2) \right) \\
&\quad + \int_{\mathbb{R}^2} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(x_2, x_1)\, \mathrm{E}\left(W^{(r)}(dx_1) W^{(s)}(dx_1) \right) \mathrm{E}\left(W^{(r)}(dx_2) W^{(s)}(dx_2) \right) \\
&= \int_{\mathbb{R}^2} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(x_1, x_2) dx_1 dx_2 \sigma^2(r) \sigma^2(s) \\
&\quad + \int_{\mathbb{R}^2} (\bar{f}_\varepsilon^{(r,s)}(x_1, x_2))' \bar{f}_\varepsilon^{(r,s)}(x_2, x_1) dx_1 dx_2 (\sigma(r, s))^2,
\end{aligned}$$

since $f_\varepsilon^{(r,s)}(x, x) = 0$ for all x and from (4.3). Then, calculations similar to (6.24) show that

$$\begin{aligned}
\mathrm{E}\, \|Q_2(C_{N,\varepsilon})(r, s)\|^2_{\mathbb{R}^{H+1}} &= \int_{\mathbb{R}^2} \|\widetilde{C}_{N,\varepsilon}(x_1, x_2, r, s)\|^2 \sigma^2(r) \sigma^2(s) dx_1 dx_2 \\
&\quad + \int_{\mathbb{R}^2} (\widetilde{C}_{N,\varepsilon}(x_1, x_2, r, s))' \widetilde{C}_{N,\varepsilon}(x_2, x_1, r, s) \sigma^2(r, s) dx_1 dx_2 \to \mathrm{E}\, \|\mathcal{I}_2(\bar{f}_\varepsilon)(r, s)\|^2_{\mathbb{R}^{H+1}},
\end{aligned} \tag{6.40}$$

where the convergence holds from calculations similar to the ones leading to (6.30) and DCT.

Proof of condition (iii): Identical calculations as in (6.23) show that

$$\mathrm{E}\,\|Q_2(C_{N,\varepsilon})(r,s)\|^2_{\mathbb{R}^{H+1}} \leqslant 2\|\widetilde{C}^{\sigma}_{N,\varepsilon}(r,s)\|^2_{L^2(\mathbb{R}^2:\mathbb{R}^{(H+1)})}. \tag{6.41}$$

Moreover, we have that, from (6.29)-(6.30)

$$\begin{aligned}
&\|\widetilde{C}^{\sigma}_{N,\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} \\
&\leqslant 8\left(\|\widetilde{C}^{\sigma}_{N,\varepsilon} - \bar{f}^{\sigma}_{\varepsilon}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} + \|\bar{f}^{\sigma}_{\varepsilon} - \bar{f}^{\sigma}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))} + \|\bar{f}^{\sigma}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))}\right) \\
&< \varepsilon + \|\bar{f}^{\sigma}\|^2_{L^2(\mathbb{R}^2:L^2(\mathbb{Y}^2:\mathbb{R}^{(H+1)}))},
\end{aligned} \tag{6.42}$$

and so $C^{\sigma}_{N,\varepsilon}(r,s)$ is $\mu\otimes\mu$- integrable. The proof of *(iii)* follows from (6.41), (6.42), and Lemma 6.2. It also completes the proof of the Lemma. □

7 Some Open Questions

The present work identifies the scaling limit of the autocovariance operator for a linear, discrete-time stochastic process taking values in a separable Hilbert space. Our focus has been on processes exhibiting long-range dependence. The tools developed herein exhibit considerable flexibility, and we conjecture that they could be adapted to identify scaling limits in a variety of other scenarios. For example:

Question 1. *In the finite-dimensional setting, it is possible to establish the convergence of higher-order statistics to Hermite processes of corresponding order; see Section 5.6 in Pipiras and Taqqu (2017). These processes can be represented as multiple Wiener-Itô stochastic integrals. We believe that the construction of the integrals in Section 4 can be extended to higher-order multiple Wiener-Itô integrals. This would pave the way for a full generalization of Theorem 5.6.3 in Pipiras and Taqqu (2017) to the setting of linear processes taking values in* L^2.

Question 2. *In the second regime, we imposed a specific representation for the coefficients* $u_j = (j+1)^{D_d - I}$ *in the process. This can easily be generalized to the case* $u_j = \ell(j)(j+1)^{D_d-I}$, *where* $\ell(j)$ *is a slowly varying function. Is it possible to relax this assumption further?*

On the other hand, our proof techniques have certain limitations. We highlight two such limitations below:

Question 3. *The "boundary" case where* $d(r) = \frac{1}{2}$ *for* $r \in A$ *with* $\mu(A) > 0$ *has not been addressed. We conjecture that when* $d(r) \in (0, 1/4]$, *the limiting random variable remains Gaussian. However, by analogy with the finite-dimensional case, we expect that the appropriate scaling operator becomes logarithmic on the set* A *where* $d(r) = \frac{1}{4}$ *for* $r \in A$.

Question 4. *In view of Corollary 3.2 and Theorem 3.3, A natural question concerns the "mixed" case where* $d(s) \in \left(0, \frac{1}{4}\right)$ *for* $s \in A$, $d(s) \in \left(\frac{1}{4}, \frac{1}{2}\right)$ *for* $s \in B$, *and* $d(s) = \frac{1}{4}$ *for* $s \in C$, *with* $\mu(A), \mu(B), \mu(C) > 0$ *and* $A \cup B \cup C = \mathbb{Y}$. *This is a challenging—yet intriguing—problem; at present, we do not even have a candidate representation for the limiting random variable in this setting.*

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References

Alvarez-Ramirez, J., Alvarez, J., Rodriguez, E., and Fernandez-Anaya, G. Time-varying Hurst exponent for US stock markets. *Physica A: Statistical Mechanics and its Applications*, 387 (24):6159–6169, 2008.

Bai, S. and Taqqu, M. S. Behavior of the generalized Rosenblatt process at extreme critical exponent values. *Annals of Probability*, 45(2):1278–1324, 2017.

Beran, J., Feng, Y., Ghosh, S., and Kulik, R. *Long-Memory Processes: Probabilistic Properties and Statistical Methods*. Springer-Verlag Berlin Heidelberg, 2013.

Bosq, D. *Linear Processes in Function Spaces: Theory and Applications*. Lecture Notes in Statistics. Springer New York, 2000.

Cajueiro, D. O. and Tabak, B. M. Testing for time-varying long-range dependence in volatility for emerging markets. *Physica A: Statistical Mechanics and its Applications*, 346(3):577–588, 2005.

Casas, I. and Gao, J. Econometric estimation in long-range dependent volatility models: Theory and practice. *Journal of Econometrics*, 147(1):72–83, 2008.

Characiejus, V. and Račkauskas, A. The central limit theorem for a sequence of random processes with space-varying long memory. *Lithuanian Mathematical Journal*, 53(2):149–160, 2013.

Characiejus, V. and Račkauskas, A. Operator self-similar processes and functional central limit theorems. *Stochastic Processes and their Applications*, 124(8):2605–2627, 2014.

Comway, J. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 1994.

Cremers, H. and Kadelka, D. On weak convergence of integral functionals of stochastic processes with applications to processes taking paths in L_p^E. *Stochastic Processes and their Applications*, 21(2):305–317, 1986.

Düker, M.-C. Limit theorems for Hilbert space-valued linear processes under long-range dependence. *Stochastic Processes and their Applications*, 128(5):1439–1465, 2018.

Düker, M.-C. Limit theorems in the context of multivariate long-range dependence. *Stochastic Processes and their Applications*, 130(9):5394–5425, 2020.

Düker, M.-C. and Zoubouloglou, P. Breuer-Major theorems for Hilbert space-valued random variables. *arXiv preprint arxiv.org/pdf/2405.11452*, 2024.

Durand, A. and Roueff, F. Hilbert space-valued fractionally integrated autoregressive moving average processes with long memory operators. *Journal of Statistical Planning and Inference*, 231:106146, 2024.

Fox, R. and Taqqu, M. S. Noncentral limit theorems for quadratic forms in random variables having long-range dependence. *Annals of Probability*, 13(2):428–446, 1985.

Fox, R. and Taqqu, M. S. Multiple stochastic integrals with dependent integrators. *Journal of Multivariate Analysis*, 21(1):105–127, 1987.

From, S. G. and Ratnasingam, S. Some new inequalities for the beta function and certain ratios of beta functions. *Results in Applied Mathematics*, 15:100302, 2022.

Giraitis, L., Koul, H., and Surgailis, D. *Large Sample Inference for Long Memory Processes.* Imperial College Press, 2012.

Horváth, L. and Kokoszka, P. Sample autocovariances of long-memory time series. *Bernoulli*, 14(2):405–418, 2008.

Hsing, T. and Eubank, R. *Theoretical Foundations of Functional Data Analysis, with an Introduction to linear operators.* Wiley Series in Probability and Statistics. John Wiley & Sons, Chichester, 2015.

Hörmann, S., Kokoszka, P., and Nisol, G. Testing for periodicity in functional time series. *Annals of Statistics*, 46(6A):2960–2984, 2018.

Jirak, M. Rate of convergence for Hilbert space valued processes. *Bernoulli*, 24(1):202–230, 2018.

Leonenko, N., Ruiz-Medina, M. D., and Taqqu, M. S. Non-central limit theorems for random fields subordinated to gamma-correlated random fields. *Bernoulli*, 23(4B):3469–3507, 2017.

Li, D., Robinson, P. M., and Shang, H. L. Long-range dependent curve time series. *Journal of the American Statistical Association*, 115(530):957–971, 2020.

Mas, A. Weak convergence for the covariance operators of a Hilbertian linear process. *Stochastic Processes and their Applications*, 99(1):117–135, 2002.

Merlevède, F., Peligrad, M., and Utev, S. Sharp conditions for the CLT of linear processes in a Hilbert space. *Journal of Theoretical Probability*, 10(3):681–693, 1997.

Muandet, K., Fukumizu, K., Sriperumbudur, B., and Schölkopf, B. Kernel mean embedding of distributions: A review and beyond. *Foundations and Trends® in Machine Learning*, 10 (1-2):1–141, 2017.

Norvaiša, R. Multiple Wiener-itô integral processes with sample paths in Banach function spaces. In Hoffmann-Jørgensen, J., Kuelbs, J., and Marcus, M. B., editors, *Probability in Banach Spaces, 9*, pages 325–347, Boston, MA, 1994. Birkhäuser Boston.

Pipiras, V. and Taqqu, M. S. *Long-Range Dependence and Self-Similarity.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

Preciado, J. and Morris, H. The varying behavior of U.S. market persistence. In *Proceedings of the World Congress on Engineering and Computer Science (WCECS 2008)*, San Francisco, USA, October 2008.

Račkauskas, A. and Suquet, C. On limit theorems for Banach-space-valued linear processes. *Lithuanian Mathematical Journal*, 50(1):71–87, 2010.

Rademacher, D., Kreiß, J.-P., and Paparoditis, E. Asymptotic normality of spectral means of Hilbert space valued random processes. *Stochastic Processes and their Applications*, 173: 104357, 2024.

Račkauskas, A. and Suquet, C. Operator fractional Brownian motion as limit of polygonal lines processes in Hilbert space. *Stochastics and Dynamics*, 11(1):49–70, 2011.

Rosenblatt, M. Some limit theorems for partial sums of quadratic forms in stationary Gaussian variables. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 49(2):125–132, 1979.

Ruiz-Medina, M. D. Spectral analysis of multifractional LRD functional time series. *Fractional Calculus and Applied Analysis*, 25(4):1426–1458, 2022.

Taqqu, M. S. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 31(4):287–302, 1975.

Tudor, C. A. Analysis of the Rosenblatt process. *ESAIM: Probability and Statistics*, 12:230–257, 2008.

Veillette, M. S. and Taqqu, M. S. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli*, 19(3):982–1005, 2013.